• SDP

$$\min_{X} \ \langle C, X\rangle \qquad \text{s.t.} \ X \succeq_{S_+} 0, \quad \mathcal{A}(X) = b$$

$X \in S^n_{+}$

$b - \mathcal{A}(X) \in \{0\}$

$$\mathcal{L}(X,\Lambda,v) = \langle C,X\rangle - \langle \Lambda, X\rangle - \langle v, b - \mathcal{A}(X)\rangle$$
$$= \langle C - \Lambda + \mathcal{A}^T(v), X\rangle - \langle b, v\rangle$$
$$= \begin{cases} -\infty & C + \mathcal{A}^T(v) - \Lambda \neq 0 \\ -\langle b,v\rangle & C + \mathcal{A}^T(v) - \Lambda = 0 \end{cases}$$

$$\max_{v} \ -\langle b, v\rangle \qquad \text{s.t.} \ C + \mathcal{A}^T(v) \succeq_{S_+} 0$$

$$\max_{v} \ -\langle b, v\rangle \qquad \text{s.t.} \ C + \mathcal{A}^T(v) \in (S^n_+)^* = S^n_+$$

• weak duality   lim value of P $\geq$ value of D

if the primal problem is limit-feasible and dual problem is feasible
then the limit value of P is bounded by value of D

let $\{X_k\}$ be a feasible sequence ($X_k \in S^n_+$ $\lim_{k\to\infty} \mathcal{A}(X) = b$)

$$\lim_{k\to\infty}\inf \langle C, X_k\rangle - -\langle b,v\rangle = \lim_{k\to\infty}\inf \langle C, X_k\rangle + \lim_{k\to+\infty}\langle \mathcal{A}(X_k), v\rangle \geq \lim_{k\to+\infty}\inf \langle \underbrace{C + \mathcal{A}^T(v)}_{S^n_+}, \underbrace{X_k}_{S^n_+}\rangle \geq 0$$

• Regular Duality   lim value of P = value of D

The dual problem is feasible has finite value d^*
$\Updownarrow$
The primal problem is limit-feasible, has finite val p^*

$\Big\}\ p^* = d^*$

$\Rightarrow$ $\Downarrow$

$\mathcal{A}^T(v) + zC \in S^n_+$
$\Downarrow$
$-\langle b,v\rangle \leq z d^*$
for $z > 0$

if D is feasible and optimal value $= d^*$
then $\forall$ v s.t. $C + \mathcal{A}^T(v) \in S^n_+$, we have $-\langle b, v\rangle \leq d^*$
then $\forall$ v $z > 0$ s.t. $C + \mathcal{A}^T(v/z) \in S^n_+$, we have $-\langle b, v/z\rangle \leq d^*$
then $\forall$ v $z > 0$ s.t. $zC + \mathcal{A}^T(v) \in S^n_+$, we have $-\langle b, v\rangle \leq z d^*$

$zC + \mathcal{A}^T(v) \in S^n_+$
$\Downarrow$
$-\langle b,v\rangle \leq z d^*$
for $z \geq 0$

also $\forall$ v s.t. $\mathcal{A}^T(v) \in S^n_+$, we have $-\langle b, v\rangle \leq 0$

if $\exists$ $\tilde{v}$ s.t. $\mathcal{A}^T(\tilde{v}) \in S^n_+$ $-\langle b, \tilde{v}\rangle > 0$

$\forall$ v feasible for D

$\mathcal{A}^T(v + t\tilde{v}) \in S^n_+$

$-\langle b, v + t\tilde{v}\rangle \to +\infty$

contradicts d^* is optimal value

$\left\{ \begin{array}{l} A^T(v) + zC \in S_+^n \\ z \geq 0 \end{array} \right\} \Rightarrow \left\{ -\langle v, b \rangle \leq z d^* \right\}$

$\therefore$ the system $\left\{ \begin{array}{l} z \geq 0 \\ A^T(v) + zC \in S_+^n \\ -\langle v, b \rangle > z d^* \end{array} \right\}$ is infeasible

$\therefore$ the system $\left\{ \begin{bmatrix} A^T & C \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v \\ z \end{bmatrix} \in \begin{bmatrix} S_+^n \\ R_+ \end{bmatrix} \quad \langle [v, z], [b, d^*] \rangle < 0 \right\}$ is infeasible

by Farkas Lemma

$\therefore$ the system $\left\{ \begin{bmatrix} A & 0 \\ C^T & 1 \end{bmatrix} \begin{bmatrix} X \\ q \end{bmatrix} = \begin{bmatrix} b \\ d^* \end{bmatrix} \quad \begin{bmatrix} X \\ q \end{bmatrix} \in \begin{bmatrix} S_+^n \\ R_+ \end{bmatrix} \right\}$ is lim-feasible

$\therefore \exists \{X_k\} \{q_k\}$ s.t. $\underbrace{\lim_{k \to +\infty} A(X_k) = b}_{\text{lim-fea}}$ $\quad \underbrace{\lim_{k \to +\infty} \langle C, X_k \rangle + q = d^*}_{\substack{\text{lim value} \leq d^* \\ \text{lim inf } \langle C, X \rangle \leq d^*}}$

by weak duality, $\forall$ feasible sequence $\{X_k\}$, $\lim_{k \to +\infty} \inf \langle C, X \rangle \geq d^*$

$\therefore \lim_{k \to +\infty} \inf \langle C, X \rangle = d^*$

$\Uparrow$

if P is lim-fea and the finite value is p^*

$\exists \{X_k\}$ s.t. $X_k \in S_+^n$ $\quad \lim_{k \to +\infty} A(X_k) = b$ $\quad \lim_{k \to \infty} \langle C, X \rangle = p^*$

Assume D is infeasible (we'll get a contradiction)

if D is infeasible, we have

$$A^T(v) + zC \in S_+^n \quad \Rightarrow \quad z \leq 0$$

for any $A^T(v) + zC \in S_+^n$ $z > 0$, v/z is feasible for D

$\therefore$ the system $\{ A^T(v) + zC \in S_+^n \quad z > 0 \}$ is infeasible

$\therefore$ the system $\left\{ \begin{bmatrix} A^T(v) & C \end{bmatrix} \begin{bmatrix} v \\ z \end{bmatrix} \in S_+^n \quad \langle [v, z][0, -1] \rangle < 0 \right\}$ is infeasible

by Farkas lemma

then the system $\left\{ \begin{bmatrix} A \\ C^T \end{bmatrix} \begin{bmatrix} X \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \quad X \in S_+^n \right\}$ is lim-fea

$\therefore \exists \{\tilde{X}_k\}$ s.t. $\lim_{k \to +\infty} A(\tilde{X}_k) = b$ $\quad \lim_{k \to +\infty} \langle C, \tilde{X} \rangle = -1$

then consider the sequence $\{X_k + t\tilde{X}_k\}$ is a feasible sequence and the value is unbounded below

$\therefore$ when P is lim-fea, D must be fea, and $p^* = d^*$ when D is fea.

- **Strong Duality**

if P is feasible, has a finite p^* and an interior point
then D is feasible, has $d^* = p^*$

$$\left.\begin{array}{l} \text{P fea + int point} \Rightarrow \lim p^* = p^* \\ \text{P is lim-fea} \Rightarrow \text{D is fea}, \ d^* = \lim p^* \end{array}\right\} p^* = d^*$$

- **Strong Duality May Not Holds**

$$\begin{array}{ll} \min\limits_{x} & 2x_{12} \\ \text{s.t.} & \begin{bmatrix} 0 & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \succeq 0 \end{array} \xrightarrow{\text{primal}} \left.\begin{array}{ll} \min\limits_{X} & \left\langle \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, X \right\rangle \\ \text{s.t.} & \left\langle \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, X \right\rangle = 0 \\ & X \succeq_{S_+} 0 \end{array}\right\} p^* = 0 \text{ when } x_{12} = x_{21}$$

$\xrightarrow{\text{dual}}$

$$\left.\begin{array}{ll} \max\limits_{v} & 0 \cdot v \\ \text{s.t.} & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + v \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \succeq_{S_+} 0 \end{array}\right\} d \text{ is infeasible}$$

P is limit feasible with $X_k = \begin{bmatrix} 1/k & -k \\ -k & k^3 \end{bmatrix} \succeq S_+^n$

$$\lim_{k \to +\infty} \left\langle \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, X \right\rangle = \lim_{k \to +\infty} \frac{1}{k} = 0$$

the limit value $= \lim\limits_{k \to \infty} \inf -2k = -\infty$

$$\left(\begin{array}{l} 1^\circ \ \text{P has no interior}, \ \lim p^* \neq p^* \\ 2^\circ \ \text{P is lim-fea, but the lim-value is not finite} \end{array} \right)$$

- **Optimality Condition**

if P is fea and has an interior-point

$$\underset{f(X^*)}{\langle C, X^* \rangle} = \underset{g(v^*, \Lambda^*)}{-\langle b, v^* \rangle} = \inf_{X \in S_+} \left[\langle C, X \rangle - \langle \Lambda^*, X \rangle - \langle v^*, b - \mathcal{A}(X) \rangle \right] \qquad \Lambda^* = C + \mathcal{A}^T(v^*)$$

$$\leq \langle C, X^* \rangle - \langle \Lambda^*, X \rangle - \langle v^*, b - \mathcal{A}(X^*) \rangle$$

$$\leq \langle C, X^* \rangle$$

optimality condition

primal fea : $X \succeq_{S^n_+} 0 \quad \mathcal{A}(X)=b$

Dual fea : $\Lambda \succeq_{S^n_+} 0$

Complementarity : $X\Lambda = 0$

gradient vanish : $\Lambda = C + \mathcal{A}^T(\nu)$

$X\Lambda = 0 \Leftrightarrow \langle X, \Lambda \rangle = 0 \quad (X \in S^n_+ \ \Lambda \in S^n_+)$

1° if $\langle X, \Lambda \rangle = 0$

$X = PP^T \quad \Lambda = QQ^T$

$$\langle X, \Lambda \rangle = tr(X\Lambda^T) = tr(PP^TQQ^T) = tr(P^TQQ^TP) = \langle P^TQ, P^TQ \rangle = 0$$

$\therefore P^TQ = 0$

$X\Lambda = PP^TQQ^T = P0Q^T = 0$

2° if $X\Lambda = 0$

then $v^T X \Lambda v = 0 \quad \forall v$

$\langle X\Lambda, vv^T \rangle = 0 \quad \forall v$

$\therefore X\Lambda = 0$

- Log-Barrier

$$\min_X \ \langle C, X \rangle \quad \text{s.t.} \ \mathcal{A}(X) = b, \ X \succeq 0$$

$$\mathcal{L}(X, \Lambda, \nu) = \langle C, X \rangle - \langle \Lambda, X \rangle - \langle \nu, b - \mathcal{A}(X) \rangle$$

$\Downarrow$

$$\left.\begin{aligned} \min_X \ & \langle C, X \rangle - \tfrac{1}{t} \log\det(X) \\ \text{s.t.} \ & \mathcal{A}(X) = b \end{aligned}\right\} \Rightarrow X^*(t)$$

$$\tilde{\mathcal{L}}(\tilde{X}, \tilde{\nu}) = t\langle C, \tilde{X} \rangle - \log\det(\tilde{X}) - \langle \nu, b - \mathcal{A}(\tilde{X}) \rangle$$

$$= \langle \tilde{X}, tC + \mathcal{A}^T(\nu) \rangle - \log\det(\tilde{X}) - \langle \nu, b \rangle$$

optimality condition: $\begin{cases} \nabla \tilde{\mathcal{L}} = tC + \mathcal{A}^T(\tilde{\nu}) - \tilde{X}^{-1} = 0 \\ \mathcal{A}(\tilde{X}) = b \quad \tilde{X} \succeq 0 \end{cases}$

$$C + \mathcal{A}^T(\tilde{\nu}/t) - \tfrac{1}{t}\tilde{X}^{-1} = 0$$

$X^*(t)$ minimize the Lagrangian

$$\mathcal{L}(X, \Lambda^*(t), \nu^*(t)) = \langle C, X \rangle - \langle \Lambda^*(t), X \rangle - \langle \nu^*(t), b - \mathcal{A}(X) \rangle$$

where $\Lambda^*(t) = \frac{1}{t} X^*(t)^{-1} \quad \nu^*(t) = \tilde{\nu}/t$

$$\begin{aligned} p^* &\geq g(\Lambda^*(t), \nu^*(t)) \\ &= \inf_X \mathcal{L}(X, \Lambda^*(t), \nu^*(t)) \\ &= \mathcal{L}(X^*(t), \Lambda^*(t), \nu^*(t)) \\ &= \langle C, X^*(t) \rangle - \langle \tfrac{1}{t} X^*(t)^{-1}, X^*(t) \rangle - \langle \nu^*(t), b - \mathcal{A}(X) \rangle \\ &= \langle C, X^*(t) \rangle - \underbrace{n/t}_{\text{duality gap}} \end{aligned}$$

$X = X^*(t)$, $\Lambda = \Lambda^*(t)$ $\quad v = v^*(t)$ satisfies

primal : $X \succeq 0$ $\quad \mathcal{A}(X) = b$

dual : $\Lambda \succeq 0$

approximate complementarity : $X\Lambda = \frac{1}{t} I$

gradient vanishes: $C - \Lambda + \mathcal{A}^T(v) = 0$

- Search Direction

$$r_t(X, \Lambda, v) = \begin{bmatrix} \left.\begin{bmatrix} \langle A_1, X\rangle - b_1 \\ \vdots \\ \langle A_m, X\rangle - b_m \end{bmatrix}\right\} \text{primal residual} \\ \left.\begin{bmatrix} X\Lambda - \frac{1}{t} I \end{bmatrix}\right\} \text{centrality residual} \\ \left.\begin{bmatrix} C - \Lambda + \mathcal{A}^T(v) \end{bmatrix}\right\} \text{dual residual} \end{bmatrix}$$

$$r_t(X + \delta X, \Lambda + \delta\Lambda, v + \delta v) \approx r_t(X, \Lambda, v) + Dr_t(X, \Lambda, v) \begin{bmatrix} \delta X \\ \delta\Lambda \\ \delta v \end{bmatrix} := 0$$

- Path-Following Method

while true:

if $C - \Lambda + \mathcal{A}^T(v) = 0$ and $\mathcal{A}(X) = b$ and $\langle \Lambda, X\rangle \le eps$

return X, Λ, v

$t = m / \langle \Lambda, X\rangle \cdot \mu$.

solve the system (log-barrier)

$$\begin{cases} C - (\Lambda + \delta\Lambda) + \mathcal{A}^T(v + \delta v) = 0 \\ \mathcal{A}(X + \delta X) = b \\ (X + \delta X)(\Lambda + \delta\Lambda) = \frac{1}{t} I \end{cases}$$

line search on residual norm to get s

$X \mathrel{+}= s \cdot \delta X$

$\Lambda \mathrel{+}= s \cdot \delta\Lambda$

$v \mathrel{+}= s \cdot \delta v$