- Semidefinite programming

$$\min \ \langle C, X\rangle$$
$$\text{s.t. } \mathcal{A}(X) = b$$
$$X \succeq_{S_+^n} 0$$

$$\mathcal{L}(X,\Lambda,V) = \langle C,X\rangle - \langle \Lambda, X\rangle - \langle V, \mathcal{A}(X)-b\rangle$$
$$= \langle C-\Lambda-\mathcal{A}^T(V), X\rangle + \langle b, V\rangle$$
$$= \begin{cases} -\infty & C-\Lambda-\mathcal{A}^T(V) \neq 0 \\ \langle b,V\rangle & C-\Lambda-\mathcal{A}^T(V) = 0 \end{cases}$$

$$\max_{\Lambda,V} \ \langle b, V\rangle$$
$$\text{s.t. } C-\Lambda-\mathcal{A}^T(V) = 0$$
$$\Lambda \succeq S_+^n$$

Weak duality: if X is primal feasible and Λ, V are dual feasible

$$\langle C,X\rangle \geq \langle b,V\rangle$$

$$\langle C,X\rangle - \langle b,V\rangle = \langle C,X\rangle - \langle \mathcal{A}(X), V\rangle = \langle C-\mathcal{A}^T(V), X\rangle = \langle \Lambda, X\rangle \geq 0$$

$$p^* = g(\Lambda^*, V^*) = \inf_X \mathcal{L}(X,\Lambda^*,V^*)$$
$$= \inf_X \langle C,X\rangle - \langle \Lambda^*, X\rangle - \langle \mathcal{A}(X), V^*\rangle$$
$$\leq \langle C,X^*\rangle - \langle \Lambda^*, X^*\rangle - \langle \mathcal{A}(X^*), V^*\rangle$$
$$\leq p^*$$

$$\langle \mathcal{A}(X), V\rangle = \sum v_i \ A_i : X$$
$$= \langle \textstyle\sum v_i A_i, X\rangle$$

eg. Strong duality of SDP may not hold

$$\min \ 2x_{12}$$
$$\text{s.t. } \begin{bmatrix} 0 & x_{12} \\ x_{12} & x_{22} \end{bmatrix} \succeq 0$$

primal is not strictly feasible

$$\xrightarrow{\text{primal}} \quad \min_X \ \langle \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{C}, X\rangle$$
$$\text{s.t. } \langle \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{A_1}, X\rangle = 0 \quad b_1$$
$$X \succeq_{S_+^n} 0$$

$x_{22} \geq 0 \quad 0 \cdot x_{22} - x_{12}^2 \geq 0$

the primal optimal is 0 (when $x_{12} = 0$)

dual

$$\max_{\Lambda,V} \ b \cdot V$$
$$\text{s.t. } \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \Lambda = V \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$
$$\Lambda \succeq_{S_+^n} 0$$

$$\begin{bmatrix} -V & 1 \\ 1 & 0 \end{bmatrix} \succeq_{S_+^n} 0$$

$-V \geq 0 \quad -V \cdot 0 - 1 \geq 0$ dual is infeasible

$$\min \ x_{11}$$
$$\text{s.t. } \begin{bmatrix} x_{11} & 1 \\ 1 & x_{22} \end{bmatrix} \succeq_{S_+^n} 0$$

$$\xrightarrow{\text{primal}} \quad \min \ \langle \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, X\rangle$$
$$\text{s.t. } \langle \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, X\rangle = 2$$
$$X \succeq_{S_+^n} 0$$

$x_{11} \geq 0 \quad x_{22} \geq 0$

$x_{11} x_{22} \geq 1$

the optimal value is 0

but not attained

dual

$$\max \ 2 \cdot V$$
$$\text{s.t. } \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & V \\ V & 0 \end{bmatrix} = \Lambda$$
$$\Lambda \geq 0$$

$$\begin{bmatrix} 1 & -V \\ -V & 0 \end{bmatrix} \geq 0$$

$0 - V^2 \geq 0$

only if $V = 0$

the dual is not strictly feasible

Dual optimal is 0, but attained

optimality condition

primal feasible. $X \succeq_{S^n_+} 0$ $\mathcal{A}(X)=b$

dual feasible: $\Lambda \succeq_{S^n_+} 0$

complementary slackness: $X\Lambda = 0$

gradient vanishes: $\mathcal{A}^T(\nu) + \Lambda = C$

$\langle \Lambda, X\rangle = 0$

$\Lambda = PP^T$ $X = QQ^T$

$0 = tr(\Lambda X) = tr(PP^TQQ^T) = tr(P^TQQ^TP) = \langle P^TQ, P^TQ\rangle$

$\therefore P^TQ = 0$

$\therefore \Lambda X = PP^TQQ^T = P0Q^T = 0$

· Barrier of Primal and Dual

$$\min_X \langle C, X\rangle \quad \text{s.t. } \mathcal{A}(X) = b,\ X \succeq_{S^n_+} 0$$

$$\max_{\Lambda,\nu} \langle b, \nu\rangle \quad \text{s.t. } \mathcal{A}^T(\nu) + \Lambda = C,\ \Lambda \succeq_{S^n_+} 0$$

$\downarrow$

$$\min_X \langle C, X\rangle - \tfrac{1}{t}\log\det(X) \quad \text{s.t. } \mathcal{A}(X) = b$$

$$\min_{\Lambda,\nu} -\langle b, \nu\rangle - \tfrac{1}{t}\log\det(\Lambda) \quad \text{s.t. } \mathcal{A}^T(\nu) + \Lambda = C$$

$$\left.\begin{aligned}&\min_X \langle C, X\rangle - \tfrac{1}{t}\log\det X\\ &\text{s.t. } \mathcal{A}(X) = b\end{aligned}\right\}\ \text{primal barrier}$$

$$\mathcal{L}(X,\nu) = \langle C, X\rangle - \tfrac{1}{t}\log\det(X) - \langle \nu, \mathcal{A}(X) - b\rangle$$

$$= \langle C - \mathcal{A}^T(\nu), X\rangle - \tfrac{1}{t}\log\det(X) + \langle \nu, b\rangle$$

$$\nabla\mathcal{L} = C - \tfrac{1}{t}X^{-1} - \mathcal{A}^T(\nu) = 0 \qquad X = \tfrac{1}{t}(C - \mathcal{A}^T(\nu))^{-1}$$

$$g(\nu) = \inf_X \mathcal{L}(X,\nu)$$

$$= \begin{cases} -\infty & \text{if } C - \mathcal{A}^T(\nu) \not\succeq_{S^n_+} 0 \\ n/t - \tfrac{1}{t}\log\det\left(\tfrac{1}{t}(C - \mathcal{A}^T(\nu))^{-1}\right) + \langle \nu, b\rangle \end{cases}$$

if $C - \mathcal{A}^T(\nu)$ has a negative eigenvalue λ_i, with eigenvector u

let $X = \alpha \cdot uu^T$

$$\mathcal{L}(X,\nu) = \langle C - \mathcal{A}^T(\nu), \alpha uu^T\rangle - \tfrac{1}{t}\log\det \alpha uu^T + \langle \nu, b\rangle \longrightarrow -\infty \text{ as } \alpha \to +\infty$$

$$\begin{aligned}&\max\ n/t - \tfrac{1}{t}\log\det\left(\tfrac{1}{t}(C - \mathcal{A}^T(\nu))^{-1}\right) + \langle \nu, b\rangle\\ &\text{s.t. } (C - \mathcal{A}^T(\nu)) \succeq_{S^n_+} 0\end{aligned} \Rightarrow \left.\begin{aligned}&\max_{\Lambda,\nu} \tfrac{1}{t}\log\det(\Lambda) + \langle \nu, b\rangle\\ &\text{s.t. } C - \mathcal{A}^T(\nu) = \Lambda\\ &\qquad \Lambda \succeq_{S^n_+} 0\end{aligned}\right\}\ \text{dual barrier}$$

KKT for primal barrier

primal feasible: $\mathcal{A}(X) = b$ $X \succeq_{S^n_+} 0$

dual feasible: $\Lambda + \mathcal{A}^T(\nu) = C$ $\Lambda \succeq_{S^n_+} 0$

gradient vanishes: $C - \mathcal{A}^T(\nu) = \tfrac{1}{t}X^{-1}$ $\Rightarrow$ $X\Lambda = \tfrac{1}{t}I$

$$\begin{aligned} \min_{\lambda, v} \quad & -\langle b, v\rangle - \tfrac{1}{t}\log\det(\lambda) \\ \text{s.t.} \quad & \mathcal{A}^T(v) + \lambda = C \end{aligned} \Bigg\} \text{ dual barrier}$$

$$\mathcal{L}(\lambda, v, X) = -\langle b, v\rangle - \tfrac{1}{t}\log\det(\lambda) + \langle X, \mathcal{A}^T(v) + \lambda - C\rangle$$

$$= \langle \mathcal{A}(X) - b, v\rangle - \tfrac{1}{t}\log\det(\lambda) + \langle \lambda, X\rangle - \langle C, X\rangle$$

$$\nabla_\lambda \mathcal{L} = -\tfrac{1}{t}\lambda^{-1} + X = 0 \qquad \lambda^{-1} = tX$$

$$g(X) = \inf_{\lambda, v} \mathcal{L}$$

$$= \begin{cases} -\infty & \mathcal{A}(X) - b \neq 0 \\ -\infty & X \not\succ_{S_{++}^n} 0 \\ -\tfrac{1}{t}\log\det(\lambda) + n/t - \langle C, X\rangle \end{cases}$$

$$\begin{aligned} \max_{X} \quad & -\tfrac{1}{t}\log\det\left(\tfrac{1}{t}X^{-1}\right) + n/t - \langle C, X\rangle \\ \text{s.t.} \quad & \mathcal{A}(X) = b \\ & X \succ_{S_{++}^n} 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} \min_{X} \quad & \langle C, X\rangle - \tfrac{1}{t}\log\det(X) \\ \text{s.t.} \quad & \mathcal{A}(X) = b \\ & X \succ_{S_{++}^n} 0 \end{aligned} \Bigg\} \text{ primal barrier}$$

primal barrier and dual barrier have the same optimality condition

if $X(t), \lambda(t), v(t)$ solve the optimality condition, then $X(t)$ solves the primal barrier, $\lambda(t), v(t)$ solves the dual barrier

KKT for dual barrier:

$$\left.\begin{aligned} &\text{primal feasible:} && \lambda \succ_{S_{++}^n} 0 \quad \mathcal{A}^T(v) + \lambda = C \\ &\text{dual feasible:} && X \succeq_{S_{++}^n} 0 \quad \mathcal{A}(X) = b \\ &\text{gradient vanishes:} && X\lambda = \tfrac{1}{t}I \end{aligned}\right\}$$

· Path-following Interior Point Method

generate $(X^k, \lambda^k, v^k) \approx (X(t^k), \lambda(t^k), v(t^k))$

key details $\begin{cases} \text{proximing to central path} \\ \text{update : newton-like step} \end{cases}$

· Local Neighborhoods of the Central Path

let $\mathcal{F}^\circ = \{(X, \lambda, v) : \mathcal{A}(X) = b,\ \mathcal{A}^T(v) + \lambda = C \quad X \succ 0 \quad \lambda \succ 0\}$ be the strictly feasible set

given $X \succ 0,\ \lambda \succ 0 \quad \mu(X, \lambda) = \langle X, \lambda\rangle / n$

if X is primal feasible and λ, v is dual feasible $\langle C, X\rangle - \langle b, v\rangle = \langle C, X\rangle - \langle \mathcal{A}(X), v\rangle = \langle X, \lambda\rangle$

$$d_F(X, \lambda) = \| \lambda(X\lambda) - \mu(X, \lambda)\cdot I \|_2$$

$$= \| X^{1/2}\lambda X^{1/2} - \mu(X, \lambda)\cdot I \|_F$$

$$= \| \lambda^{1/2} X \lambda^{1/2} - \mu(X, \lambda) I \|_F$$

(measure of proximity of $X\lambda$ and $\mu(X, \lambda) I$

$$\mathcal{N}_F(\theta) = \{ (X, \lambda, v) \in \mathcal{F}^\circ : d_F(X, \lambda) \le \theta \mu(X, \lambda) \}$$

- Newton Step

$$\begin{bmatrix} \mathcal{A}^T(v) + \Lambda - C \\ \mathcal{A}(X) - b \\ X\Lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{t}I \end{bmatrix} \qquad X, \Lambda \succ 0$$

$\downarrow$ natural newton step

$$\begin{bmatrix} 0 & \mathcal{A}^T & I \\ \mathcal{A} & 0 & 0 \\ \Lambda & 0 & X \end{bmatrix} \begin{bmatrix} \Delta X \\ \Delta v \\ \Delta\Lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{t}I - X\Lambda \end{bmatrix}$$

ΔX and $\Delta\Lambda$ may not be symmetric
($\frac{1}{t}I - X\Lambda$ may not be symmetric)

- Search Direction

$$r_t(X, \Lambda, v) = \begin{bmatrix} \mathcal{A}^T(v) + \Lambda - C \\ \mathcal{A}X - b \\ X\Lambda - \frac{1}{t}I \end{bmatrix} = \begin{bmatrix} \left[\sum_i v_i A_i + \Lambda - C\right] \\ \begin{bmatrix} \langle A_1, X\rangle - b_1 \\ \vdots \\ \langle A_m, X_m\rangle - b_m \end{bmatrix} \\ \left[X\Lambda - \frac{1}{t}I\right] \end{bmatrix} \begin{matrix} \} \; r_d \text{ dual residual} \\ \} \; r_p \text{ primal residual} \\ \} \; r_c \text{ centrality residual} \end{matrix}$$

$$r_t(X + \Delta X, \Lambda + \Delta\Lambda, v + \Delta v) \approx r_t(X, \Lambda, v) + Dr_t \cdot \begin{bmatrix} \Delta X \\ \Delta\Lambda \\ \Delta v \end{bmatrix} := 0$$

$$\begin{array}{c} ① \\ ② \\ ③ \end{array} \begin{bmatrix} 0 & \mathcal{A}^T(\cdot) & I \\ \mathcal{A}(\cdot) & 0 & 0 \\ \mathcal{E}(\cdot) & 0 & \mathcal{F}(\cdot) \end{bmatrix} \begin{bmatrix} \Delta X \\ \Delta v \\ \Delta\Lambda \end{bmatrix} = -r_t(X, \Lambda, v) = -\begin{bmatrix} r_d \\ r_p \\ r_c \end{bmatrix}$$

$$\Delta X = \mathcal{E}^{-1}\left(-r_c - \mathcal{F}(\Delta\Lambda)\right) \qquad \text{from (3)}$$

$$= \mathcal{E}^{-1}\left[-r_c - \mathcal{F}(-r_d - \mathcal{A}^T(\Delta v))\right] \qquad \text{from (1)}$$

$$\mathcal{A}\left(\mathcal{E}^{-1}\left[-r_c - \mathcal{F}(-r_d - \mathcal{A}^T(\Delta v))\right]\right) = -r_p \qquad \text{from (2)}$$

$$\mathcal{A}\mathcal{E}^{-1}(-r_c - \mathcal{F}(-r_d)) + \mathcal{A}\mathcal{E}^{-1}\mathcal{F}\mathcal{A}^T(\Delta v) = -r_p$$

$$\Delta v = (\mathcal{A}\mathcal{E}^{-1}\mathcal{F}\mathcal{A}^T)^{-1}\left[-r_p + \mathcal{A}\mathcal{E}^{-1}(r_c - \mathcal{F}(r_d))\right] \qquad \text{solve for } \Delta v$$

$$\Delta\Lambda = -r_d - \mathcal{A}^T(\Delta v) \qquad \text{solve for } \Delta\Lambda$$

$$\Delta X = \mathcal{E}^{-1}(-r_c - \mathcal{F}(\Delta\Lambda)) \qquad \text{solve for } \Delta X$$

· Path Following Method

$$\begin{aligned} \min.\ & \langle C,X\rangle \\ \text{s.t.}\ & \mathcal{A}(X)=b \\ & X \succeq_{S_+} 0 \end{aligned} \Longrightarrow \begin{aligned} \min\ & t\langle C,X\rangle - \log\det(X) \\ \text{s.t.}\ & \mathcal{A}(X)=b \end{aligned}$$

$$\tilde{\mathcal{L}}(\tilde{X},\tilde{V}) = t\langle C,\tilde{X}\rangle - \log\det(\tilde{X}) - \langle \tilde{V}, \mathcal{A}(X)-b\rangle$$

$$\nabla_X \tilde{\mathcal{L}} = tC - \tilde{X}^{-1} - \mathcal{A}^T(\tilde{V}) := 0$$

$$C - \underbrace{\tfrac{1}{t}\tilde{X}^{-1}}_{\Lambda} - \mathcal{A}^T(\tilde{V}/t) = 0 \quad = \nabla_X \mathcal{L} = \frac{d}{dX}\langle C,X\rangle - \langle \Lambda, X\rangle - \langle \tilde{V}/t, \mathcal{A}(X)-b\rangle$$

if $\tilde{X},\tilde{V}$ solves the log-barrier, then $\tilde{X}$ minimize the Lagrangian

$$\mathcal{L}(X,\Lambda,V) = \langle C,X\rangle - \langle \Lambda, X\rangle - \langle V, \mathcal{A}(X)-b\rangle$$

where $\Lambda = \frac{1}{t}\tilde{X}^{-1}$, $V = \tilde{V}/t$

$$\begin{aligned} p^* \geq g(\Lambda,V) &= \inf_X \mathcal{L}(X,\Lambda,V) \\ &= \langle C,\tilde{X}\rangle - \langle \Lambda,\tilde{X}\rangle - \langle V, \mathcal{A}(\tilde{X})-b\rangle \\ &= \langle C,\tilde{X}\rangle - \tfrac{1}{t}\langle \tilde{X}^{-1}\tilde{X}\rangle \\ &= \langle C,\tilde{X}\rangle - \underbrace{n/t}_{\text{duality gap}} \end{aligned}$$

when X is primal feasible, Λ, V is dual feasible

$\langle \Lambda, X\rangle = n/t$ is the duality gap

$t = n/\langle \Lambda, X\rangle$

while true:

if $\mathcal{A}^T(V) + \Lambda - C = 0$ and $\mathcal{A}(X) = b$ and $\langle \Lambda, X\rangle \leq eps$

return X, Λ, V

$t = n/\langle \Lambda, X\rangle \cdot u$ // u controls the advancement of central path

solve the system (solve the log-barrier)

$$\begin{cases} \mathcal{A}^T(V+\Delta V) + (\Lambda+\Delta\Lambda) - C = 0 \\ \mathcal{A}(X+\Delta X) - b = 0 \\ (X+\Delta X)(\Lambda+\Delta\Lambda) = \frac{1}{t}I \end{cases}$$

line search to get s

$X \mathrel{+}= s\cdot\Delta X$

$\Lambda \mathrel{+}= s\cdot\Delta\Lambda$

$V \mathrel{+}= s\cdot\Delta\Lambda$