· Linear Program

$$\min_x \; c^T x \quad \text{s.t.} \; Ax = b, \; x \geq 0$$

$$\begin{aligned} \mathcal{L}(x,\lambda,v) &= c^T x - \lambda^T x - v^T(Ax-b) \\ &= (c-\lambda-A^T v)^T x + b^T v \\ &= \begin{cases} -\infty & c-\lambda-A^T(v) \neq 0 \\ b^T v & c-\lambda-A^T v = 0 \end{cases} \end{aligned}$$

$$\max_{\lambda, v} \; b^T v \quad \text{s.t.} \; c-\lambda-A^T v = 0, \; \lambda \geq 0$$

weak duality. if x is primal feasible and λ, v are dual feasible

$$c^T x \geq b^T v$$

$$c^T x - b^T v = c^T x - (Ax)^T v = x^T(c - A^T v) = x^T \lambda \geq 0$$

$$\begin{aligned} p^* = g(\lambda^*, v^*) &= \inf_x \mathcal{L}(x, \lambda^*, v^*) \\ &= \inf_x \; c^T x - \lambda^{*T} x - v^{*T}(Ax-b) \\ &\leq c^T x^* - \lambda^{*T} x^* - v^{*T}(Ax^*-b) \\ &\leq p^* \end{aligned}$$

optimality condition

- primal feasible: $x \geq 0 \quad Ax = b$
- dual feasible: $\lambda \geq 0$
- complementary slackness: $\lambda_i x_i = 0$
- gradient vanishes: $A^T v + \lambda = c$

Simplex method: maintain equality conditions and aim for $x \geq 0 \;\; \lambda \geq 0$

Interior point method: maintain linear conditions and aim for complementarity

· Barrier of primal and dual

$$\min_x \; c^T x \quad \text{s.t.} \; Ax = b, \; x \geq 0$$

$$\downarrow$$

$$\min_x \; c^T x - \tfrac{1}{t}\sum \log(x) \quad \text{s.t.} \; Ax = b$$

$$\min_{\lambda, v} \; -b^T v \quad \text{s.t.} \; A^T v + \lambda = c, \; \lambda \geq 0$$

$$\downarrow$$

$$\min_{\lambda, v} \; -b^T v - \tfrac{1}{t}\sum \log \lambda \quad \text{s.t.} \; A^T v + \lambda = c$$

$\rightarrow$ dual $\leftarrow$

the primal barrier problem and dual barrier problem are dual of each other

$$\min_x \; c^T x - \frac{1}{t}\sum \log(x)$$ primal barrier

$$\mathcal{L}(x,v) = c^T x - \frac{1}{t}\sum \log(x) - v^T(Ax-b)$$
$$= (c - A^T v)^T x - \frac{1}{t}\sum \log x + b^T v$$
$$\nabla \mathcal{L} = c - A^T v - \frac{1}{t}\frac{1}{x} = 0 \qquad x = \frac{1}{t}\left(\frac{1}{c - A^T v}\right)$$
$$g(v) = \inf_x \mathcal{L}(x,v)$$
$$= \begin{cases} -\infty & (c - A^T v)_i \le 0 \;\; \forall i \\ n/t - \frac{1}{t}\sum \log\left(\frac{1}{t}\frac{1}{c - A^T v}\right) + b^T v \end{cases}$$

dual barrier

$$\max_v \; n/t - \frac{1}{t}\sum \log\left(\frac{1}{t}\frac{1}{c - A^T v}\right) + b^T v \quad \Rightarrow \quad \max_{v,\lambda} \; \frac{1}{t}\sum \log(\lambda) + b^T v$$
$$\text{s.t.} \quad c - A^T v > 0 \qquad\qquad c - A^T v = \lambda, \;\; \lambda > 0$$

KKT for primal barrier

primal feasible: $Ax = b \quad x > 0$

dual feasible: $\lambda + A^T v = c \quad \lambda > 0$

gradient vanishes: $c - A^T v = \frac{1}{t}\frac{1}{x} \quad \Rightarrow \quad \lambda_i x_i = \frac{1}{t}$

$$\left.\begin{aligned} &\min_{\lambda, v} \; -b^T v - \frac{1}{t}\sum \log \lambda \\ &\text{s.t.} \;\; A^T v + \lambda = c \end{aligned}\right\} \text{ dual barrier}$$

$$\mathcal{L}(\lambda, v, x) = -b^T v - \frac{1}{t}\sum \log \lambda + x^T(A^T v + \lambda - c)$$
$$= (Ax - b)^T v - \frac{1}{t}\sum \log \lambda + x^T \lambda - c^T x$$
$$\nabla_\lambda \mathcal{L} = -\frac{1}{t}\frac{1}{\lambda} + x = 0 \qquad \lambda = \frac{1}{tx}$$
$$g(x) = \inf_{\lambda, v} \mathcal{L}$$
$$= \begin{cases} -\infty & \text{if } Ax - b \ne 0 \\ -\infty & x_i \le 0 \;\; \forall i \\ -\frac{1}{t}\sum \log \frac{1}{tx} + n/t - c^T x \end{cases}$$

$$\begin{aligned} &\max \; -c^T x - \frac{1}{t}\sum \log \frac{1}{tx} \\ &\text{s.t.} \;\; Ax = b, \; x > 0 \end{aligned} \quad \Rightarrow \quad \left.\begin{aligned} &\min \; c^T x - \frac{1}{t}\sum \log x \\ &\text{s.t.} \;\; Ax = b, \; x > 0 \end{aligned}\right\} \text{ primal barrier}$$

KKT for dual barrier:

primal feasible: $\lambda > 0 \quad A^T v + \lambda = c$

dual feasible: $x > 0 \quad Ax = b$

gradient vanishes: $\lambda_i x_i = \frac{1}{t}$

primal barrier and dual barrier have the same optimality condition

assume primal problem and dual problem are strictly feasible,

→ the primal dual central path is $\{(x(t), v(t), \lambda(t))\}$

s.t. $x(t)$ solves the primal barrier

$\lambda(t), v(t)$ solve the dual barrier

$\Updownarrow$

$Ax(t) = b \quad x(t) > 0$

$A^T v(t) + \lambda(t) = c \quad \lambda(t) > 0$

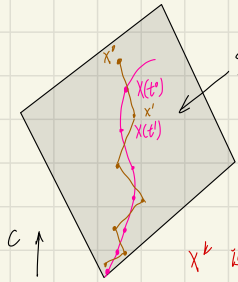
$x_i(t)\,\lambda_i(t) = \frac{1}{t}$

for the primal barrier, obj is strictly convex

for the dual barrier, obj is strictly convex, A is full-rank

$\therefore \; (x(t), \lambda(t), v(t))$ is unique

· Path-following Interior-point method



$\{x : Ax = b \ \ x > 0\}$

generate $(x^k, \lambda^k, v^k) \approx (x(t^k), \lambda(t^k), v(t^k))$

key details $\begin{cases} \cdot \text{ proximity to central path} \\ \cdot \text{ increase } t \\ \cdot \text{ update } (x^k, \lambda^k, v^k) \end{cases}$

x^k is the approximation of $x(t^k)$ as $t^k \to \infty$

solutions to the primal-barrier

· Neighborhoods of Central path

let $\mathcal{F}^0 = \{(x, \lambda, v) : Ax = b \ \ A^Tv + \lambda = c \ \ x > 0 \ \ \lambda > 0\}$ be the feasible set

$$\mathcal{F}^0 \longmapsto \mathbb{R}^n_{++}$$

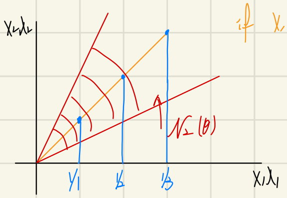
$$(x, \lambda, v) \qquad \begin{pmatrix} x_1\lambda_1 \\ \vdots \\ x_n\lambda_n \end{pmatrix}$$

given $x, \lambda \in \mathbb{R}^n_+$ $\quad \mu(x, \lambda) = \frac{x^T\lambda}{n}$

if x is primal feasible and λ v is dual feasible $c^Tx - b^Tv = x^T\lambda$ $\quad$ $n\mu(x,\lambda)$ is the duality gap.

two norm neighborhood: for $\theta \in (0,1)$

$$\mathcal{N}_2(\theta) = \left\{ (x, \lambda, v) : (x, \lambda, v) \in \mathcal{F}^0, \ \| x \cdot s - \mu(x,s)\mathbf{1} \|_2 \le \theta \mu(x,s) \right\}$$



if x, λ is on the central path

$x_i\lambda_i = \frac{1}{t} \ \ \forall i$

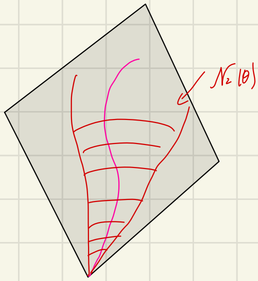
let $x = \lambda_1 x_1$

$y = \lambda_2 x_2$

$K = (x, y) : \left\| \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} \frac{x+y}{2} \\ \frac{x+y}{2} \end{pmatrix} \right\|_2 \le \theta \cdot \frac{x+y}{2}$

if $(x_1, y_1) \in K$ $\quad (x_2, y_2) \in K$

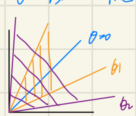
$a(x_1, y_1) + b(x_2, y_2) = (ax_1 + bx_2, ay_1 + by_2) \in K$



if $\theta = 0$ $\quad x_i\lambda_i = \mu(x, \lambda) = \frac{x^T\lambda}{n}$

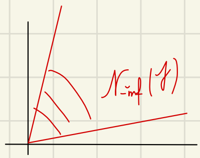
then $x \cdot \lambda = \frac{1}{t}\mathbf{1}$ for some t, x, λ is on the central path

the larger θ is, the bigger neighborhood is



one-sided infinity-norm neighborhood for $\gamma \in [0,1)$

$$\mathcal{N}_{-\inf}(\gamma) = \{ (x,\lambda,v) : (x,\lambda,v) \in \mathcal{F}^\circ ,\ x_i\lambda_i \geq \gamma\,\mu(x,\lambda) \}$$



if $\gamma \approx 0$ the $\mathcal{N}_{-\inf}(\gamma) \approx \mathcal{F}^\circ$

· Newton Step

$$\begin{bmatrix} A^T v + \lambda - c \\ Ax - b \\ \mathrm{diag}(x)\,\mathrm{diag}(\lambda) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{t}\mathbf{1} \end{bmatrix} \qquad x \geq 0 \quad \lambda \geq 0$$

$$\begin{bmatrix} 0 & A^T & I \\ A & 0 & 0 \\ \mathrm{diag}(\lambda) & 0 & \mathrm{diag}(x) \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta v \\ \Delta \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{t}\mathbf{1} - x\cdot\lambda \end{bmatrix}$$

Short-Step Path-Following Algorithm (2-norm neighborhood, small)

chose $\theta, \delta \in [0,1)$ s.t. $\dfrac{\theta^2+\delta^2}{2^{3/2}(1-\theta)} \leq \left(1-\dfrac{\delta}{\sqrt{n}}\right)\theta$ (can chose $\theta = \delta = 0.4$)

let $(x^0, \lambda^0, v^0) \in \mathcal{N}_2(\theta)$

for $k = 0, 1 \dots$

let $\frac{1}{t} = \left(1 - \frac{\delta}{\sqrt{n}}\right)\mu(x,\lambda)$

compute newton step $\begin{bmatrix} 0 & A^T & I \\ A & 0 & 0 \\ S & 0 & X \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta v \\ \Delta \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{t}\mathbf{1} - x_i\lambda_i \end{bmatrix}$

$(x^{k+1}, \lambda^{k+1}, v^{k+1}) = (x^k, \lambda^k, v^k) + (\Delta x, \Delta\lambda, \Delta v)$

the sequence generated by SPF satisfies

$(x^k, \lambda^k, v^k) \in \mathcal{N}_2(\theta)$

$\mu(x^{k+1}, \lambda^{k+1}) = \left(1 - \frac{\delta}{\sqrt{n}}\right)\mu(x^k, s^k)$

$$x^{k+1}\cdot\lambda^{k+1} = (x^k + \Delta x)(\lambda^k + \Delta\lambda) = x^k\lambda^k + x^k\Delta\lambda + \lambda^k\Delta x + \Delta x\Delta\lambda$$
$$= x^k\lambda^k + \tfrac{1}{t}\mathbf{1} - x^k\lambda^k + \Delta x\Delta\lambda$$
$$= \tfrac{1}{t}\mathbf{1} + \Delta x\Delta\lambda$$
$$\mu(x^{k+1}, \lambda^{k+1}) = \frac{1}{n}(x^{k+1})^T(\lambda^{k+1}) = \frac{1}{n}\left(\frac{n}{t} + \Delta x^T\Delta\lambda\right)$$
$$= \frac{1}{t} + \mu(\Delta x, \Delta\lambda)$$
$$= \left(1 - \frac{\delta}{\sqrt{n}}\right)\mu(x^k, \lambda^k) + \underbrace{\mu(\Delta x, \Delta\lambda)}_{0}$$

$$\begin{cases} A^T\Delta y + \Delta s = 0 \\ A\Delta x = 0 \end{cases}$$
$$\underbrace{\Delta x^T A^T\Delta y}_{0} + \Delta x^T\Delta s = 0$$
$$\therefore \Delta x^T\Delta s = 0$$

theoretically fast, practically slow

· Long-Step path Following Algorithm (inf - norm neighborhood, large)

choose $\gamma \in (0,1)$ $\quad 0 < \sigma_{min} < \sigma_{max} < 1$

let $(x^0, \lambda^0, v^0) \in \mathcal{N}_{-\infty}(\gamma)$

for $k = 0, 1 \cdots$

choose $\sigma \in [\sigma_{min}, \sigma_{max}]$, $t = \sigma u(x^k, \lambda^k)$

compute newton step
$$\begin{bmatrix} 0 & A^T & I \\ A & 0 & 0 \\ S & 0 & X \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta v \\ \Delta \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ tI - X\lambda \end{bmatrix}$$

chose $\alpha_k \in [0,1]$ s.t.
$$(x^k, \lambda^k, v^k) + \alpha(\Delta x, \Delta \lambda, \Delta v) \in \mathcal{N}_{-inf}(\gamma)$$
$$(x^{k+1}, \lambda^{k+1}, v^{k+1}) = (x^k, \lambda^k, v^k) + \alpha(\Delta x, \Delta \lambda, \Delta v)$$

the sequence generated by LPF satisfies

$(x^k, \lambda^k, v^k) \in \mathcal{N}_{-inf}(\gamma)$

$u(x^{k+1}, \lambda^{k+1}) \leq (1 - \frac{\delta}{n}) u(x^k, \lambda^k)$ $\quad$ δ depends on γ, σ_{min} and σ_{max}

LPF is practically faster than SPF

· Infeasible Interim-point Algorithm

given $x^0 > 0$ $\quad \lambda^0 > 0$

given $\gamma \in (0,1)$, $\beta \geq 1$

$$\begin{bmatrix} A^T v + \lambda - c \\ Ax - b \\ \lambda x \end{bmatrix} = \begin{bmatrix} r_d \\ r_p \\ r_c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ tI \end{bmatrix}$$

$$\mathcal{N}_{-inf}(\gamma) = \left\{ (x, \lambda, v) : \ \|(r_p, r_c)\| \leq \frac{\beta u}{u^0} \|r_p^0, r_c^0\| \quad x > 0 \quad \lambda > 0, \ x_i \lambda_i \geq \gamma u_i \right\}$$

chose $\gamma \in (0,1)$ $\quad 0 < \sigma_{min} < \sigma_{max} < 0.5$ and $\beta \geq 1$

chose (x^0, λ^0, v^0) with $x^0, \lambda^0 > 0$

For $k = 0, 1 \cdots$

chose $\sigma \in [\sigma_{min}, \sigma_{max}]$

$t = \sigma \cdot u(x^k, s^k)$

Solve newton step
$$\begin{bmatrix} 0 & A^T & I \\ A & 0 & 0 \\ S & 0 & X \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta v \\ \Delta \lambda \end{bmatrix} = \begin{bmatrix} -r_d \\ -r_p \\ tI - X\lambda \end{bmatrix}.$$

chose $\alpha \in [0,1]$ s.t
$$(x^k, \lambda^k, v^k) + \alpha(\Delta x, \Delta \lambda, \Delta v) \in \mathcal{N}_{-inf}(\gamma, \beta)$$
$$u(x^k + \alpha \Delta x, \lambda + \alpha \Delta \lambda) \leq (1 - 0.01\alpha) \ u(x_k, \lambda_k)$$