- **Kronecker Product**

the kronecker product of matrix $A \in \mathbb{R}^{p \cdot q}$ with $B \in \mathbb{R}^{rs}$ is

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1q}B \\ \vdots & & \vdots \\ a_{p1}B & \cdots & a_{pq}B \end{bmatrix}$$

the Vec-operator for matrix $A \in \mathbb{R}^{m \times n}$ is

$$vec(A) = (a_{11} \cdots a_{m1} \quad a_{12} \cdots a_{m2} \quad \cdots \quad a_{1n} \cdots a_{mn})$$

$$tr(A^T B) = vec(A)^T vec(B)$$

- **Properties of Kronecker Product**

$$(\alpha A) \otimes B = A \otimes (\alpha B) = \alpha (A \otimes B)$$

$$(A \otimes B)^T = A^T \otimes B^T$$

$$(A + B) \otimes C = A \otimes C + B \otimes C$$

$$A \otimes (B + C) = A \otimes B + A \otimes C$$

$$(A \otimes B) \otimes C = A \otimes (B \otimes C)$$

$\forall \; A \in \mathbb{R}^{m \times n} \quad B \in \mathbb{R}^{p \times q} \quad C \in \mathbb{R}^{r \times s}$

$$A \otimes (B \otimes C) = \begin{bmatrix} a_{11}[B \otimes C] & \cdots & a_{1n}[B \otimes C] \\ \vdots & & \\ a_{m1}[B \otimes C] & \cdots & a_{mn}[B \otimes C] \end{bmatrix} \qquad (A \otimes B) \otimes C = \begin{bmatrix} (A \otimes B)_{11} C & \cdots & (A \otimes B)_{mp,nq} C \\ & & \\ & & \end{bmatrix}$$

$$[(A \otimes B) \otimes C]_{ij} = (A \otimes B)[(i-1)//r+1, (j-1)//s+1] \cdot C[i\%r, j\%s]$$

$$= A[((i-1)//r+1-1)//p+1, ((j-1)//s+1-1)//q+1] \cdot B[((i-1)//r+1)\%p, ((j-1)//s+1)\%q] \; C[i\%r, j\%s]$$

$$= A[(i-1)//rp+1, (j-1)//sq+1] \; B[((i-1)//r+1)\%p, ((j-1)//s+1)\%q] \; C[i\%r, j\%s]$$

$$[A \otimes (B \otimes C)]_{ij} = A[(i-1)//rp+1, (j-1)//sq+1] \cdot (B \otimes C)[i\%pr, j\%qs]$$

$$= A[(i-1)//rp+1, (j-1)//sq+1] \cdot B[(i\%pr-1)//r+1, (j\%qs-1)//s+1] \cdot C[(i\%pr)\%r, (j\%qs)\%s]$$

$$= A[(i-1)//rp+1, (j-1)//sq+1] \cdot B[(i\%pr-1)//r+1, (j\%qs-1)//s+1] \cdot C[i\%r, j\%s]$$

when $i = apr + 1 \qquad 0 \le a < m$

$$((i-1)//r+1)\%p = (ap+1)\%p = 1$$

$$(i\%pr-1)//r+1 = 1$$

when $i = apr + br + 1 \quad 0 < b < p$

$$\left.\begin{aligned} ((i-1)//r+1)\%p &= (ap+b+1)\%p = (b+1)\%p \\ (i\%pr-1)//r+1 &= b+1 \end{aligned}\right\} = \begin{cases} b+1 & \text{if } b<p-1 \\ 0=p & \text{if } b=p-1 \end{cases}$$

when $i = apr + br + c \quad 0 < c < r$

$$((i-1)//r+1)\%p = (ap+b+1)\%p = (b+1)\%p$$

$$(i\%pr-1)//r+1 \quad (br+c-1)//r+1 = b+1$$

$(A\otimes B)(C\otimes D) = AC\otimes BD$

$\forall A \in R^{p\times q} \quad B\in R^{r\times s}$

$C \in R^{q\times k} \quad D \in R^{s\times l}$

$$(\overset{p\times k}{AC} \otimes \overset{r\times l}{BD})_{ij} = AC[(i-1)//r+1, (j-1)//l+1]\cdot BD[i\%r, j\%l]$$

$$= \left[\sum_{n_1=1}^{q} A[(i-1)//r+1, n_1]\, C[n_1, (j-1)//l+1]\right]\left[\sum_{n_2=1}^{s} B[i\%r, n_2]\cdot D[n_2, j\%l]\right]$$

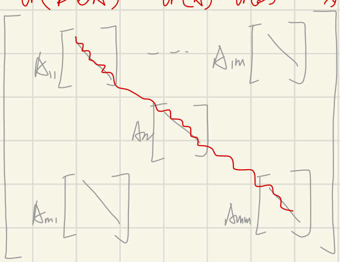
$$[(A\otimes B)(C\otimes D)]_{ij} = \sum_{n=1}^{qs} (A\otimes B)[i,n]\cdot(C\otimes D)[n,j]$$

$$= \sum_{n=1}^{qs} A[(i-1)//r+1, (n-1)//s+1]\cdot B[i\%r, n\%s]\cdot C[(n-1)//s+1, (j-1)//l+1]\cdot D[n\%s, j\%l]$$

$$= \sum_{n=1}^{qs} A[(i-1)//r+1, (n-1)//s+1]\cdot C[(n-1)//s+1, (j-1)//l+1]\cdot B[i\%r, n\%s]\, D[n\%s, j\%l]$$

$$= \sum_{n_1=1}^{q} A[(i-1)//r+1, n_1]\, C[n_1, (j-1)//l+1]\cdot \sum_{n_2=1}^{s} B[i\%r, n_2]\, D[n_2, j\%l]$$

$tr(A\otimes B) = tr(B\otimes A) = tr(A)\cdot tr(B) \quad \forall A\in R^{m\times m} \quad B\in R^{n\times n}$



$$tr(A\otimes B) = \sum_i A_{ii}\, tr(B)$$

$$= tr(A)\cdot tr(B)$$

$$= tr(B)\cdot tr(A)$$

$$= tr(B\otimes A)$$

$det(A\otimes B) = det(B\otimes A) = det(A)^n\cdot det(B)^m \quad \forall A\in R^{m\times m} \quad B\in R^{n\times n}$

$A\otimes B$ is nonsingular iff A and B are nonsingular if $A \in \mathbb{R}^{m \times m}$ $B \in \mathbb{R}^{n \times n}$ are non-singular, $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$

$$
\begin{aligned}
\left[(A \otimes B)(A^{-1} \otimes B^{-1})\right]_{ij} &= \sum_{k=1}^{n^2} (A \otimes B)_{ik} \, (A^{-1} \otimes B^{-1})_{kj} \\
&= \sum_{k=1}^{n^2} A[(i-1)//n+1, (k-1)//n+1] \, B[i\%n, k\%n] \\
&\qquad A^{-1}[(k-1)//n+1, (j-1)//n+1] \cdot B^{-1}[k\%n, j\%n] \\
&= \sum_{p=1}^{n} A[(i-1)//n+1, p] \, A^{-1}[p, (j-1)//n+1] \cdot \sum_{q=1}^{n} B[i\%n, q\%n] \, B^{-1}[q\%n, j\%n] \\
&= I[(i-1)//n+1, (j-1)//n+1] \cdot I[i\%n, j\%n] \\
&= \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}
\end{aligned}
$$

· Kronecker Sum

the kronecker sum of $A \in \mathbb{R}^{m\times m}$ $B \in \mathbb{R}^{n\times n}$ is

$$A \oplus B = (I_n \otimes A) + (B \otimes I_m)$$

$$\underbrace{\begin{bmatrix} [A] & [0] \\ [0] & [A] \end{bmatrix}}_{I_n \otimes A} + \underbrace{\begin{bmatrix} [b_{11} I] & [b_{12} I] \\ [b_{n1} I] & [b_{nn} I] \end{bmatrix}}_{B \otimes I_m}$$

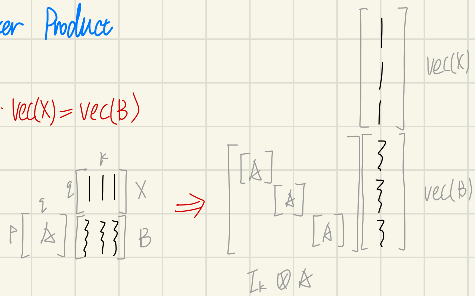
$$(A \oplus B) \otimes C \neq (A \otimes C) \oplus (B \otimes C)$$

$$A \otimes (B \oplus C) \neq (A \otimes B) \oplus (A \otimes C)$$

· Matrix Equations and Kronecker Product

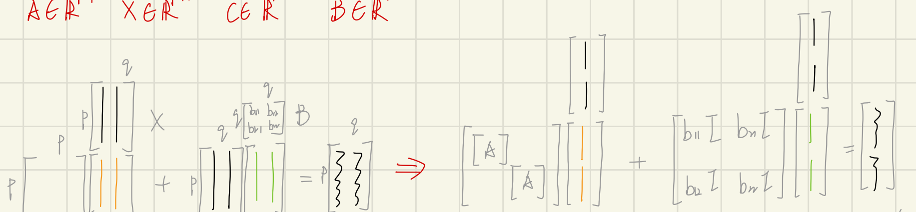
$$AX = B \quad \Rightarrow \quad (I_k \otimes A) \cdot vec(X) = vec(B)$$

$A \in \mathbb{R}^{p\times q}$ $X \in \mathbb{R}^{q\times k}$



$$AX + XB = C \quad \Rightarrow \quad \left[(I_q \otimes A) + (B^T \otimes I_p)\right] vec(X) = vec(C) \quad \Rightarrow \quad (A \oplus B^T)\, vec(X) = vec(C)$$

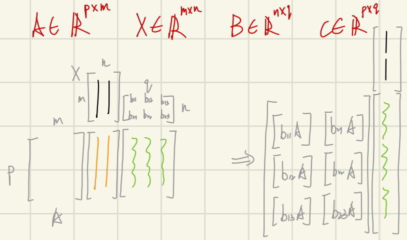
$A \in \mathbb{R}^{p\times p}$ $X \in \mathbb{R}^{p\times q}$ $C \in \mathbb{R}^{p\times q}$ $B \in \mathbb{R}^{q\times q}$



$$(I_q \otimes A)\, vec(X) + (B^T \otimes I_p)\, vec(X) = vec(C)$$

$$AXB = C \quad \Rightarrow \quad (B^T \otimes A)\, vec(X) = vec(C)$$

$A \in \mathbb{R}^{p\times m}$ $X \in \mathbb{R}^{m\times n}$ $B \in \mathbb{R}^{n\times q}$ $C \in \mathbb{R}^{p\times q}$



· The Symmetric Kronecker Product

for any $S \in S^n$, $svec(S) = (S_{11}, \sqrt{2}S_{21} \cdots \sqrt{2}S_{n1},\ S_{22}, \sqrt{2}S_{32} \cdots)$

$$vec\left(\begin{bmatrix} S_{11} & & & \\ S_{21} & S_{22} & & \\ \vdots & & \ddots & \\ S_{n1} & \cdots & \cdots & S_{nn} \end{bmatrix} .* \begin{bmatrix} 1 & & & \\ \sqrt{2} & 1 & & \\ \vdots & \vdots & \ddots & \\ \sqrt{2} & \sqrt{2} & \cdots & 1 \end{bmatrix} \right)$$

$$trace(ST) = sum(S .* T) = svec(S)^T svec(T) \quad \forall S, T \in S^n$$

(1 for diagonal elements, 2 for off-diag elements)

$\exists\ QQ^T = I$ s.t. $Q\, vec(S) = svec(S)$, $\quad Q^T svec(S) = vec(S)$

$$\begin{bmatrix} S_{11} & S_{12} & \cdots & S_{1n} \\ \vdots & & & \vdots \\ S_{n1} & \cdots & \cdots & S_{nn} \end{bmatrix} .* \begin{bmatrix} 1 & & & \\ \sqrt{2} & 1 & & \\ \sqrt{2} & \sqrt{2} & 1 & \\ \vdots & \vdots & & \ddots \\ \sqrt{2} & \sqrt{2} & \cdots & 1 \end{bmatrix}$$

$$Q = \begin{bmatrix} \text{——} & vec\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) & \text{——} \\ \text{——} & vec\left(\begin{bmatrix} 0 & \sqrt{2}/2 & 0 \\ \sqrt{2}/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) & \text{——} \\ \text{——} & vec\left(\begin{bmatrix} 0 & 0 & \sqrt{2}/2 \\ 0 & 0 & 0 \\ \sqrt{2}/2 & 0 & 0 \end{bmatrix}\right) & \text{——} \\ \text{——} & vec\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) & \text{——} \\ & \vdots & \end{bmatrix} \quad \tfrac{1}{2}(n+1)n \qquad QQ^T = I$$

$\longrightarrow n^2 \longleftarrow$

let $Q \in \mathbb{R}^{[\frac{1}{2}(n+1)n \times n^2]}$ be a mapping matrix for $S \in S^n$

the symmetric kronecker product is defined as

$$G \otimes_s H = \tfrac{1}{2} Q (G \otimes H + H \otimes G) Q^T \quad \forall\ G, H \in \mathbb{R}^{n \times n}$$

$$\begin{aligned} (G \otimes_s H)\, svec(U) &= \tfrac{1}{2} Q (G \otimes H + H \otimes G) Q^T svec(U) \\ &= \tfrac{1}{2} Q (G \otimes H + H \otimes G)\, vec(U) \\ &= \tfrac{1}{2} Q \left[(G \otimes H)\, vec(U) + (H \otimes G)\, vec(U) \right] \\ &= \tfrac{1}{2} Q \left[vec(HUG^T) + vec(GUH^T) \right] \\ &= \tfrac{1}{2} Q\, vec(HUG^T + GUH^T) \\ &= \tfrac{1}{2} svec(HUG^T + GUH^T) \end{aligned}$$

$(G \otimes_s H)\, svec(U) = \tfrac{1}{2} svec\left[(HUG^T) + (HUG^T)^T \right]$ the implicit definition of symmetric kronecker product

- Properties of Symmetric Kronecker Product

$$(\alpha A)\otimes_s B = A\otimes_s(\alpha B) = \alpha\cdot(A\otimes_s B) \qquad \forall A,B\in\mathbb{R}^{n\times n}$$

$$(\alpha A)\otimes_s B = \frac{1}{2}Q\left[(\alpha A)\otimes B + B\otimes(\alpha A)\right]Q^T = \alpha\cdot A\otimes_s B$$

$$(A\otimes_s B)^T = A^T\otimes_s B^T \quad \forall A,B\in\mathbb{R}^{n\times n} \qquad (A\otimes_s B)^* = A^*\otimes_s B^* \quad \forall A,B\in\mathbb{C}^{n\times n}$$

$$\begin{aligned}(A\otimes_s B)^T &= \frac{1}{2}\left[Q(A\otimes B + B\otimes A)Q^T\right]^T\\ &= \frac{1}{2}Q\left[(A\otimes B)^T + (B\otimes A)^T\right]Q\\ &= \frac{1}{2}Q\left[A^T\otimes B^T + B^T\otimes A^T\right]Q\\ &= A^T\otimes_s B^T\end{aligned}$$

$A\otimes_s I$ is symmetric iff A is symmetric

$$(A+B)\otimes_s C = (A\otimes_s C) + (B\otimes_s C) \qquad \forall A\ B\ C \in\mathbb{R}^{n\times n}$$

$$A\otimes_s(B+C) = A\otimes_s B + A\otimes_s C$$

$$\begin{aligned}(A+B)\otimes_s C &= \frac{1}{2}Q\left[(A+B)\otimes C + C\otimes(A+B)\right]Q^T\\ &= \frac{1}{2}Q(A\otimes C + C\otimes A)Q^T + \frac{1}{2}Q(B\otimes C + C\otimes B)Q^T\\ &= A\otimes_s C + B\otimes_s C\end{aligned}$$

$$(A\otimes_s B)(C\otimes_s D) = \frac{1}{2}\left[AC\otimes_s BD + AD\otimes_s BC\right] \qquad \forall A\ B\ C\ D\in\mathbb{R}^{n\times n}$$

$$\begin{aligned}(A\otimes_s B)(C\otimes_s D)[i,j] &= \sum_k (A\otimes_s B)[i,k]\,(C\otimes_s D)[k,j]\\ &= \frac{1}{4}\sum_k \left[Q(A\otimes B + B\otimes A)Q^T\right]_{ik}\left[Q(C\otimes D + D\otimes C)Q^T\right]_{kj}\\ &= \frac{1}{4}\sum_k \left[Q[i,:]^T(A\otimes B + B\otimes A)Q^T[:,k]\right]\left[Q[k,:](C\otimes D + D\otimes C)Q^T[:,j]\right]\\ &= \frac{1}{4}Q[i,:]^T(A\otimes B + B\otimes A)\left[\sum_k Q^T[:,k]\,Q[k,:]^T\right](C\otimes D + D\otimes C)\,Q^T[:,j]\\ &= \frac{1}{4}Q[i,:]^T(A\otimes B + B\otimes A)(C\otimes D + D\otimes C)\,Q^T[:,j]\\ &= \frac{1}{4}Q[i,:]^T\left[(A\otimes B)(C\otimes D) + (A\otimes B)(D\otimes C) + (B\otimes A)(C\otimes D) + (B\otimes A)(D\otimes C)\right]Q^T[:,j]\\ &= \frac{1}{4}Q[i,:]^T\left[\underset{\sim}{AC\otimes BD} + AD\otimes BC + BC\otimes AD + \underset{\sim}{BD\otimes AC}\right]Q^T[:,j]\\ &= \frac{1}{4}Q[i,:]^T(AC\otimes BD + BD\otimes AC)Q^T[:,j] + \frac{1}{4}Q[i,:]^T(AD\otimes BC + BC\otimes AD)Q^T[:,j]\\ &= \frac{1}{2}(AC\otimes_s BD + AD\otimes_s BC)\end{aligned}$$

$(A \otimes_s A)^{-1} = A^{-1} \otimes_s A^{-1} \quad \forall A \in \mathbb{R}^{n\times n}$ is nonsingular

$$(A\otimes_s A)(A^{-1}\otimes_s A^{-1}) = \frac{1}{2}\left[AA^{-1}\otimes_s AA^{-1} + AA^{-1}\otimes_s AA^{-1}\right]$$
$$= \frac{1}{2}\left[I\otimes_s I + I\otimes_s I\right]$$
$$= I\otimes_s I$$
$$= \frac{1}{2} Q\left[I\otimes I + I\otimes I\right]Q^T$$
$$= I$$

$(A\otimes_s B)^{-1} \neq A^{-1}\otimes_s B^{-1}$ in general

• Solving for AHO Direction

$$\begin{cases} \mathcal{A}(X) = b \\ \mathcal{A}^T(y) + \Lambda = C \\ X\Lambda = \frac{1}{t} I \end{cases} \Rightarrow \begin{cases} \mathcal{A}(X) = b \\ \mathcal{A}^T(v) + \Lambda = C \\ \frac{1}{2}(X\Lambda + \Lambda X) = \frac{1}{t} I \end{cases} \Rightarrow \begin{cases} \mathcal{A}(\Delta X) = b - \mathcal{A}(X) = -r_p \\ \mathcal{A}^T(\Delta v) + \Delta\Lambda = C - \mathcal{A}^T(v) - \Lambda = -r_d \\ \frac{1}{2}(\Delta X\Lambda + \Lambda\Delta X + X\Delta\Lambda + \Delta\Lambda X) = \frac{1}{t}I - \frac{1}{2}(X\Lambda + \Lambda X) = -r_c \end{cases}$$

central path — AHO direction — newton step

$$\begin{bmatrix} 0 & \mathcal{A}(\cdot) & 0 \\ \mathcal{A}^T(\cdot) & 0 & I \\ 0 & \mathcal{E}(\cdot) & \mathcal{f}(\cdot) \end{bmatrix} \begin{bmatrix} \Delta v \\ \Delta X \\ \Delta\Lambda \end{bmatrix} = \begin{bmatrix} -r_p \\ -r_d \\ -r_c \end{bmatrix}$$

where
$$\mathcal{E}(\Delta X) = \frac{1}{2}(\Delta X\Lambda + \Lambda\Delta X)$$
$$f(\Delta\Lambda) = \frac{1}{2}(\Delta\Lambda X + X\Delta\Lambda)$$

$$AX + XB = C \Rightarrow (I\otimes A + B^T\otimes I)\,\mathrm{vec}(X) = \mathrm{vec}(C)$$

$$\begin{bmatrix} A & \\ & A \end{bmatrix} + \begin{bmatrix} B_{11}I & B_{21}I \\ B_{12}I & B_{22}I \end{bmatrix}$$

$$\mathcal{E}(\Delta X) + f(\Delta\Lambda) = \frac{1}{2}(\Lambda\Delta X + \Delta X\Lambda) + \frac{1}{2}(X\Delta\Lambda + \Delta\Lambda X) = -r_c$$
$$\frac{1}{2}(I\otimes\Lambda + \Lambda\otimes I)\,\mathrm{vec}(\Delta X) + \frac{1}{2}(I\otimes X + X\otimes I)\,\mathrm{vec}(\Delta\Lambda) = \mathrm{vec}(-r_c)$$
$$\frac{1}{2}Q(I\otimes\Lambda + \Lambda\otimes I)Q^T Q\,\mathrm{vec}(\Delta X) + \frac{1}{2}Q(I\otimes X + X\otimes I)Q^T Q\,\mathrm{vec}(\Delta\Lambda) = Q\,\mathrm{vec}(-r_c)$$

$$(I\otimes_s\Lambda)\cdot\mathrm{svec}(\Delta X) + (I\otimes_s X)\,\mathrm{svec}(\Delta\Lambda) = \mathrm{svec}(-r_c)$$

$$\mathcal{A}^T(\Delta v) + \Delta\Lambda = -r_d$$
$$\mathrm{vec}(\mathcal{A}^T(\Delta v)) + \mathrm{vec}(\Delta\Lambda) = \mathrm{vec}(-r_d)$$
$$Q\,\mathrm{vec}(\mathcal{A}^T(\Delta v)) + \mathrm{svec}(\Delta\Lambda) = \mathrm{svec}(-r_d)$$
$$Q\,\mathrm{vec}(\mathcal{A}^T(\Delta v)) = Q\,\mathrm{vec}\Big(\sum_i A_i\Delta v_i\Big) = Q\cdot\sum_i \Delta v_i\cdot\mathrm{vec}(A_i) = \sum_i \Delta v_i\cdot Q\,\mathrm{vec}(A_i) = \sum_i \Delta v_i\,\mathrm{svec}(A_i)$$
$$\sum_i \Delta v_i\,\mathrm{svec}(A_i) + \mathrm{svec}(\Delta\Lambda) = \mathrm{svec}(-r_d)$$

$\mathcal{A}(\Delta X) = -r_p$

$$\langle A_i, \Delta X \rangle = -r_{p_i}$$

$$vec(A_i)^T vec(\Delta X) = -r_{p_i}$$

$$vec(A_i)^T Q^T Q\, vec(\Delta X) = -r_{p_i}$$

$$(Q\, vec(A_i))^T\ svec(\Delta X) = -r_{p_i}$$

$$svec(A_i)^T\ svec(\Delta X) = -r_{p_i}$$

$$\begin{bmatrix} 0 & svec(A[:])^T & 0 \\ svec(A[:]) & 0 & I \\ & I \otimes_s \Lambda & I \otimes_s X \end{bmatrix} \begin{bmatrix} \Delta V \\ svec(\Delta X) \\ svec(\Lambda) \end{bmatrix} = \begin{bmatrix} -r_p \\ svec(-r_d) \\ svec(-r_c) \end{bmatrix} \qquad svec(A[:]) = \begin{bmatrix} svec(A_1) & \cdots & svec(A_k) \end{bmatrix}$$

$$r_{p_i} = \langle A_i, X \rangle - b_i = vec(A_i)^T vec(X) - b_i$$
$$= vec(A_i)\, Q^T Q\, vec(X) - b_i$$
$$= svec(A_i)^T svec(X) - b_i$$
$$r_p = svec(A[:])^T\, svec(X) - b$$

$$r_d = \mathcal{A}^T(V) + \Lambda - C = \sum_i \Delta V_i A_i + \Lambda - C$$
$$vec(r_d) = \sum_i V_i\, vec(A_i) + vec(\Lambda) - vec(C)$$
$$svec(r_d) = \sum_i V_i\, svec(A_i) + svec(\Lambda) - svec(C)$$
$$svec(r_d) = svec(A[:])V + svec(\Lambda) - svec(C)$$

$$r_c = \frac{1}{2}(X\Lambda + \Lambda X) - \frac{1}{t} I$$

- Block Elimination

$$svec(\Delta X) = (I \otimes_s \Lambda)^{-1}\left(svec(-r_c) - (I \otimes_s X)\, svec(\Lambda) \right) \qquad \text{from (3)}$$
$$= (I \otimes_s \Lambda)^{-1}\left[svec(-r_c) - (I \otimes_s X)\left(svec(-r_d) - svec(A[:])\Delta V \right) \right] \qquad \text{from (2)}$$

$$svec(A[:])^T\ (I \otimes_s \Lambda)^{-1}\left[svec(-r_c) - (I \otimes_s X)\left(svec(-r_d) - svec(A[:])\Delta V \right) \right] = -r_p$$

$$svec(A[:])^T (I \otimes_s \Lambda)^{-1}\left[svec(-r_c) - (I \otimes_s X)\, svec(-r_d) \right] + r_p = -svec(A[:])^T (I \otimes_s \Lambda)^{-1} (I \otimes_s X)\, svec(A[:])\Delta V$$

$$\mathrm{svec}(\Delta\Lambda) = \mathrm{svec}(-r_d) - \mathrm{svec}(A[:])\Delta v$$
$$\mathrm{svec}(\Delta X) = (I \otimes_s \Lambda)^{-1}\left[\mathrm{svec}(-r_c) - (I \otimes_s X)\,\mathrm{svec}(\Delta\Lambda)\right]$$

· Line Search

$$\begin{bmatrix} 0 & \mathrm{svec}(A[:])^T & 0 \\ \mathrm{svec}(A[:]) & 0 & I \\ & I \otimes_s \Lambda & I \otimes_s X \end{bmatrix} \begin{bmatrix} \Delta v \\ \mathrm{svec}(\Delta X) \\ \mathrm{svec}(\Lambda) \end{bmatrix} = \begin{bmatrix} -r_p \\ \mathrm{svec}(-r_d) \\ \mathrm{svec}(-r_c) \end{bmatrix}$$

let $y = (v, X, \Lambda)$ $\quad r(y) = (r_p, r_d, r_c)$

1° $\Lambda + s\cdot\Delta\Lambda \in S^n_{++}$

2° $X + s\cdot\Delta X \in S^n_{++}$

3° $$\frac{d}{ds}\|r(y+s\Delta y)\|_2^2\Big|_{s=0} = 2\langle r(y), Dr(y)\cdot\Delta y\rangle = 2\langle r(y), -r(y)\rangle = -2\|r(y)\|_2^2$$

$$\left.\begin{aligned}&\frac{d}{ds}\|r(y+s\Delta y)\|_2^2\Big|_{s=0} = -2\|r(y)\|_2^2 \\ &\frac{d}{ds}\|r(y+s\Delta y)\|_2^2\Big|_{s=0} = 2\|r(y)\|_2 \cdot \frac{d}{ds}\|r(y+s\Delta y)\|_2\Big|_{s=0}\end{aligned}\right\} \frac{d}{ds}\|r(y+s\Delta y)\|_2 = -\|r(y)\|_2$$