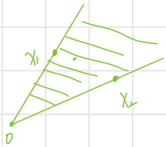
· Closed Convex Cones

2022.7.11

a convex cone is the set of all conic combinations of points in the set
(any convex cone has to be a cvx set)



Let K be a nonempty closed set, K is a closed convex cone if

$1°$ $\forall x \in K$ and $\lambda \geq 0$, $\lambda x \in K$ (K is a cone)

$2°$ $\forall x \in K, y \in K$, $x+y \in K$

} convexity

S_n^+ is a closed convex cone

Let K, L be closed convex cones, then $\{(x,y) : x \in K, y \in L\}$ is a closed convex cone

the Ice-Cream Cone $\nabla_n = \{(x,r) \in \mathbb{R}^{n-1} \times \mathbb{R} : \|x\| \leq r\}$ is a closed convex cone

if $(x_1, r_1) \in \nabla_n$ $(x_2, r_2) \in \nabla_n$

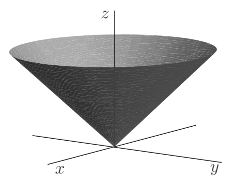
$\|x_1 + x_2\| \leq \|x_1\| + \|x_2\| = r_1 + r_2$

$\therefore (x_1+x_2, r_1+r_2) \in \nabla_n$

$\forall \lambda \geq 0$ $\|\lambda x\| = \lambda \|x\| = \lambda r$

$\therefore (\lambda x, \lambda r) \in \nabla_n$

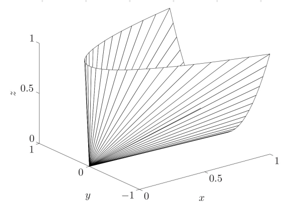
$\therefore \nabla_n$ is a closed convex cone



psd cone: the toppled Ice cream cone

$\triangleleft = \left\{ (x,y,z) : \begin{pmatrix} x & z \\ z & y \end{pmatrix} \succeq 0 \right\}$

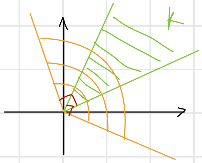
$(x \geq 0 \quad y \geq 0 \quad xy \geq z^2)$



· Dual cones

let K be a closed convex cone, the dual cone of K is

$$K^* = \{ y : \langle y, x \rangle \geq 0 \ \ \forall x \in K \}$$



the dual cone of a closed convex cone is also a closed convex cone

if $y_1 \in K^*$ $y_2 \in K^*$

$\langle y_1 + y_2, x \rangle = \langle y_1, x \rangle + \langle y_2, x \rangle \geq 0 \quad \forall x \in K$

S_+^n is self-dual

if $Y \in S_+^n$, then $Y \in (S_+^n)^*$

$\forall X \in S_+^n, \quad \langle X, Y \rangle = \langle \sum_i \lambda_i u_i u_i^T, Y \rangle$

$= \sum_i \lambda_i u_i^T Y u_i$

≥ 0

$\therefore \forall X \in S_+^n, \ \langle X, Y \rangle \geq 0 \qquad \therefore Y \in (S_+^n)^*$

if $Y \notin S_n^+$, then $Y \notin (S_+^n)^*$

$\exists q$ s.t. $q^T Y q < 0$

let $X = t q q^T \in S_+^n$

$\langle X, Y \rangle = t q^T Y q \to -\infty$ when $t \to +\infty$

$\therefore \exists X \in S_+^n \quad \langle X, Y \rangle < 0$

$\therefore Y \notin (S_+^n)^*$

the dual cone of $\triangleleft = \{ (x, y, z) : \begin{bmatrix} x & z \\ z & y \end{bmatrix} \succeq 0 \}$ is

$$\triangleleft^* = \{ (x, y, z) : \begin{bmatrix} x, & z/2 \\ z/2 & y \end{bmatrix} \succeq 0 \}$$

1° if $v \in \triangleleft^*$, then $\langle v, u \rangle \geq 0 \ \forall u \in \triangleleft$

let $v = (v_x, v_y, v_z) \in \triangleleft^*$

$\forall u = (u_x, u_y, u_z) \in \triangleleft$

$\langle u, v \rangle = u_x \cdot v_x + u_y v_y + u_z v_z$

$\geq 2\sqrt{u_x u_y v_x v_y} + u_z v_z$

$\geq 2\sqrt{u_z^2 \cdot \frac{v_z^2}{4}} + u_z v_z$

$= |u_z \cdot v_z| + u_z v_z \quad \geq 0$

$\therefore v \in \triangleleft^*$

2° if $v \notin \triangleleft^*$.

if $v_x < 0$, then $u = (1, 0, 0) \in \triangleleft$, $\langle u, v \rangle < 0$

if $v_y < 0$, then $u = (0, 0, 1) \in \triangleleft$, $\langle u, v \rangle < 0$

if $v_x \geq 0$ $v_y \geq 0$, $v_x \cdot v_y \leq v_z^2/4$

choose $u = (v_y \ \ v_x, \ -\sqrt{v_x v_y}) \in \triangleleft$

$\langle u, v \rangle = v_x v_y + v_x v_y - v_z \sqrt{v_x v_y}$

$\leq 2 v_x v_y - 2\sqrt{v_x v_y}\sqrt{v_x v_y}$

$= 0$

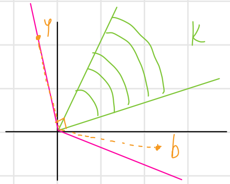
let K, L be closed convex cones, then

$$(K \times L)^* = K^* \times L^*$$

· A Separation Theorem for Closed Convex Cone

let $K \subseteq V$ be a closed convex cone, and $b \in V \setminus K$

$\underbrace{\exists\, y \in V \text{ s.t. } \langle y, x\rangle \geq 0 \ \ \forall x \in K,}_{\exists\, y \in K^*}$ and $\langle y, b\rangle < 0$



if $K = R^2_{++}$

then $K^* = R^2_+$

let $b = [1, 0] \in V \setminus K$

$\forall\, y \in K^*$, $\langle y, b\rangle \geq 0$

(can let b on the boundary of K when K is open)

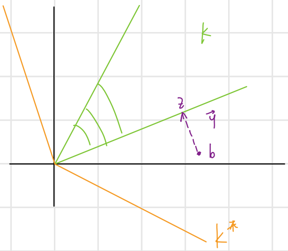
let z be the point in K that's nearest to b

let $y = z - b$

for $z = 0$ $\quad \langle y, z\rangle = 0$

for $z \neq 0$ $\quad$ if $\langle y, z\rangle \neq 0$

(then move z along a ray $\{tz : t > 0\}$ can get a point closer to b)



assume $\langle y, z\rangle > 0$, $\quad$ set $z' = (1-\alpha) z$

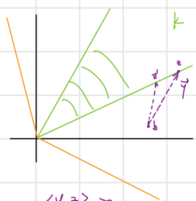
$\|z' - b\|_2^2 = \langle z - \alpha z - b, z - \alpha z - b\rangle = \langle y - \alpha z, y - \alpha z\rangle = \|y\|_2^2 + \alpha^2 \|z\|_2^2 - 2\alpha \langle y, z\rangle$

for small α, $2\alpha\langle y, z\rangle > \alpha^2 \|z\|_2^2$

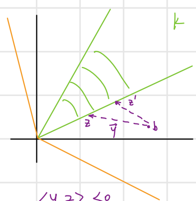
$\therefore \|z' - b\|_2^2 = \|y\|_2^2 + \alpha^2\|z\|_2^2 - \alpha^2 \|z\|_2^2 = \|y\|_2^2$

contradict with that z is the nearest point

$\therefore \langle y, z\rangle = 0$



$\langle y, z\rangle > 0$



$\langle y, z\rangle < 0$

$\forall\, x \in K \quad \langle z - b, x - z\rangle \geq 0$

if $\langle z - b, x - z\rangle < 0$

let $u = z + \alpha(x - z)$

$\|u - b\|_2^2 = \| z + \alpha(x - z) - b\|_2^2$

$= \|z - b\|_2^2 + \alpha^2 \|x - z\|_2^2 + 2\alpha\langle x - z, z - b\rangle$

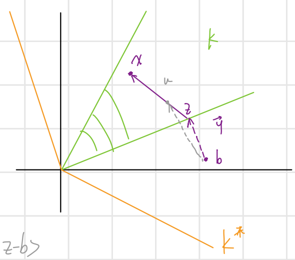
$< \|z - b\|_2^2$ $\quad$ for small α

contradict with that z is the closest in K to b

$\therefore \langle z - b, x - z\rangle \geq 0$

$= \langle y, x\rangle - \langle y, z\rangle = \langle y, x\rangle \geq 0$

$\therefore y \in K^*$



$\therefore \exists\, y = z - b$ $\quad$ s.t. $\quad y \in K^*$, $\quad \langle y, b\rangle = \langle y, b\rangle - \langle y, z\rangle = -\langle y, y\rangle < 0$

let $K \subseteq V$ be a closed convex cone, then $K^{**} = K$

$K^{**} = \{ y \mid \langle y, x \rangle \geq 0 \ \forall x \in K^* \}$

1° $b \in K \rightarrow b \in K^{**}$

$\forall b \in K$ $\quad K^* = \{ u \mid \langle u, v \rangle \geq 0 \ \forall v \in K \}$

$\therefore \forall u \in K^*, \quad \langle u, b \rangle \geq 0$

$\therefore b \in K^{**}$

2° $b \notin K \rightarrow b \notin K^{**}$

$\exists y \in K^*$ s.t. $\langle y, b \rangle < 0$ } Separation theorem

$\therefore b \notin K^{**}$

· The Farkas Lemma

let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$

exactly one of 2 systems $\begin{bmatrix} Ax = b & x \geq 0 \\ A^T y \geq 0 & b^T y < 0 \end{bmatrix}$ has a solution

proof 1

let $K = \{ Ax \mid x \in \mathbb{R}^n_+ \} \subseteq \mathbb{R}^m$

1° if $Ax = b$, $x \geq 0$ is infeasible

then $b \in V \setminus K$,

$\therefore$ by separation theorem

$\therefore \exists y \in K^* = \{ v \mid v^T A x \geq 0 \ \forall x \in \mathbb{R}^n_+ \} = \mathbb{R}^n_+$ s.t. $y^T b < 0$

$\therefore \exists y$ s.t. $y^T A x \geq 0 \ \forall x \in \mathbb{R}^n_+$ $\quad y^T b < 0$

$\therefore A^T y \in (\mathbb{R}^n_+)^* = \mathbb{R}^n_+ \quad y^T b < 0$

$\therefore A^T y \geq 0 \quad y^T b < 0$ is feasible

if $Ax = b$ $x \geq 0$ is infeasible

then $A^T y \geq 0$ $b^T y < 0$

use Farkas Lemma on closed convex cone $\{ Ax \mid x \geq 0 \}$

2° If both of $\begin{bmatrix} Ax = b & x \geq 0 \\ A^T y \geq 0 & b^T y < 0 \end{bmatrix}$ is feasible

then let y be a solution of $\{ A^T y \geq 0, b^T y < 0 \}$

$\left.\begin{array}{l} y^T A x = (A^T y)^T x \geq 0 \\ y^T b < 0 \end{array}\right\}$ a conflict

$\begin{bmatrix} Ax = b & x \geq 0 \\ A^T y \geq 0 & b^T y < 0 \end{bmatrix}$ can not both be feasible

- **Adjoint Operators**

let $A: V \mapsto W$ be a linear operator, $A^T: W \mapsto V$ is an adjoint of A if

$$\langle y, A(x)\rangle = \langle A^T(y), x\rangle \quad \forall\ x \in V \text{ and } y \in V$$

let $V = S^n$ $W = R^m$. $A(x) = (\langle A_1, x\rangle, \cdots, \langle A_m, x\rangle)$, then $A^T(y) = \sum_{i=1}^{m} y_i A_i$

$$\langle A(x), y\rangle = \sum_i y_i \langle A_i, x\rangle = \langle \sum_i y_i A_i, x\rangle$$

Analogy:

let $V_1, V_2, \cdots V_n$ and $W_1, \cdots W_m$ be vector spaces

let $A_{ij}: V_i \to W_i$ be linear operators

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \qquad\qquad\qquad\qquad \begin{bmatrix} y \end{bmatrix}$$

$$\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2m} \\ \vdots & \vdots & & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{bmatrix} \begin{bmatrix} A(x) \end{bmatrix} = \begin{bmatrix} A_{11}^T & A_{21}^T & \cdots & A_{m1}^T \\ A_{12}^T & A_{22}^T & & \vdots \\ \vdots & \vdots & & \vdots \\ A_{1n}^T & A_{2n}^T & \cdots & A_{mn}^T \end{bmatrix} \begin{bmatrix} A^T y \end{bmatrix}$$

$$\begin{bmatrix} y^T \end{bmatrix} \qquad\qquad\qquad\qquad \begin{bmatrix} x^T \end{bmatrix}$$

- **Farkas Lemma, Cone Version**

let $K \subseteq V$ be a closed convex cone, $A: V \mapsto V$ be a linear operator

$C = \{A(x) \mid x \in K\}$, the closure of C, $\bar{C}$, is a closed convex cone

closure of C is the set containing all points in C and limiting points

let $K \subseteq V$ be a closed convex cone, the system $\begin{cases} A(x) = b \\ x \in K \end{cases}$ is limit-feasible if

$$\begin{cases} \exists \text{ a sequence } \{x_1 \cdots x_{+\infty}\} \text{ s.t. } x_k \in K \\ \lim_{k \to +\infty} A(x_k) = b \end{cases}$$

$A(x) = b$ $x \in K$ is limit feasible $\Leftrightarrow$ $b \in$ closure of $\{A(x) \mid x \in K\}$

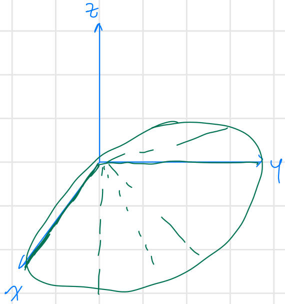
let $K \subseteq V$ be a close convex cone, $b \in W$, $A: V \mapsto W$ is a linear operator

either $\begin{cases} A(x) = b \quad x \in K \quad \text{is limit-feasible} \\ A^T(y) \in K^* \quad \langle b, y\rangle < 0 \quad \text{is feasible} \end{cases}$

eg. why limit feasible

let $k = \lhd = \left\{ (x,y,z) \,\middle|\, \begin{bmatrix} x & z \\ z & y \end{bmatrix} \succeq 0 \right\}$ is a closed convex cone

the projection of k on yz plane is $0 \cup (\underset{y}{\mathbb{R}_+} \times \underset{z}{\mathbb{R}})$

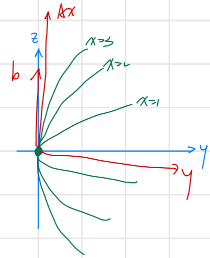


when $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is not a closed convex cone

if $Ax = b$ $x \in \lhd$ is infeasible

then $b \in V \setminus \{ Ax \mid x \in k \}$

$\exists$ y s.t. $y \in \underbrace{\{ Ax \mid x \in \lhd \}^*}_{\text{not even a closed convex cone}}$ st. $\langle y, b \rangle < 0$



A^Ty on the yz plane $\{ Ax = b \quad x \in k \}$ when b on z axis,

$\{ \langle x, A^Ty \rangle \geq 0 \ \forall x \in k \quad b^Ty < 0 \}$

when b on z axis, $\{ Ax = b, x \in k \}$ is infeasible

$\{ \langle x, A^Ty \rangle \geq 0 \ \forall x \in k, \ b^Ty < 0 \}$ is infeasible.

eg. when b pointing upwards, y must point downwards.

can chose Ax very near z axis,

so that $\langle Ax, y \rangle = \langle x, A^Ty \rangle < 0$

Proof of Farkas lemma Cone Version

either $\begin{cases} A(x) = b \quad x \in k \quad \text{is limit-feasible} \\ A^T(y) \in k^* \quad \langle b, y \rangle < 0 \quad \text{is feasible} \end{cases}$

1° if $A(x) = b$ $x \in k$ is limit feasible

$\exists \{ x_k : k = 1, \cdots +\infty, x_k \in k \}$ $\lim_{k \to +\infty} A(x_k) = b$

$\langle y, b \rangle = \langle y, \lim_{k \to +\infty} A(x_k) \rangle = \lim_{k \to +\infty} \langle y, A(x_k) \rangle = \lim_{k \to +\infty} \langle A^T(y), x_k \rangle$

if $A^T(y) \in k^*$, then $\langle y, b \rangle \geq 0$

$\therefore$ $A^T(y) \in k^*$, $\langle y, b \rangle < 0$ is infeasible

2° if $A(x) = b$ $x \in k$ is not limit-feasible

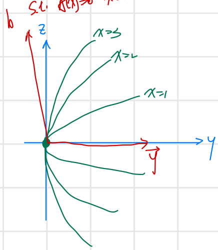
let $C = \{ A(x) \mid x \in k \}$

$\bar{C}$, the closure of C, is a closed convex cone, and $b \notin \bar{C}$

$\therefore \exists \underbrace{y \in \bar{C}^*}_{\Updownarrow}$ s.t. $\langle y, b \rangle < 0$

$\langle y, A(x) \rangle \geq 0 \ \forall x \in k \iff \langle A^T(y), x \rangle \geq 0 \ x \in k \iff A^T(y) \in k^*$

$\therefore$ $A^T(y) \in k^*$, $\langle y, b \rangle < 0$ is feasible



eg. Farkas Lemma for S_+^n

let $K = S_+^n \in \mathbb{R}^{n\times n}$ $\quad W \in \mathbb{R}^m$ $\quad A(x) = (\langle A_1, x\rangle, \cdots \langle A_m, x\rangle) \in \mathbb{R}^m$

$\langle y, A(x)\rangle = \sum_i y_i \langle A_i, x\rangle = \langle x, \sum_i y_i A_i\rangle = \langle A^T(y), x\rangle$

if $A(x)=b$ $\ x\in S_+^n$ is limit feasible

$\exists \{x_k \ \ k=1,\cdots,+\infty,\ x_k\in S_+^n\}$ s.t. $\lim_{k\to+\infty} A(x_k) = b$

$\langle y,b\rangle = \langle y, \lim_{k\to+\infty} A(x_k)\rangle = \lim_{k\to+\infty} \langle y, A(x_k)\rangle = \lim_{k\to+\infty} \langle A^T(y), x_k\rangle$

if $A^T(y) \in S_+^{n*} = S_+^n$,

then $\langle A^T(y), x\rangle \geq 0$

$\therefore \{A^T(y)\in S_+^n,\ \langle y,b\rangle<0\}$ is infeasible

if $A(x)=b$, $x\in S_+^n$ is not lim-fea

then $C=\{A(x) \mid x\in S_+^n\}$, $\bar{C}$ is a closed convex cone, $b\notin \bar{C}$

$\exists$ $y\in \bar{C}^*$, s.t. $\langle y,b\rangle<0$

$\Updownarrow$

$\langle y, A(x)\rangle \geq 0 \ \ \forall x\in S_+^n$, $\ \therefore A^T(y)\in S_+^n$

$\therefore \{A^T(y)\in S_+^n,\ \langle y,b\rangle<0$ is feasible $\}$

· Cone Programming

let $K\subseteq V$ $\ L\subseteq W$ be closed convex cones, $b\in W$, $c\in V$ $\quad A: V\mapsto W$ be a linear operator

A cone program is

$$\min_{x} \ \langle c,x\rangle$$
$$\text{s.t.} \ \ b-A(x)\in L$$
$$x\in K$$

for $L=\{0\}$, we get cone programs in equational form, $A(x)=b$

eg.

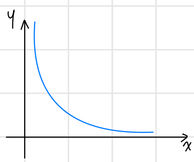
$$\min_{(x,y,z)} \ x$$
$$\text{s.t.} \ \ z=1$$
$$(x,y,z)\in \triangle$$

$\Rightarrow$

$$\min_{X} \left\langle \begin{bmatrix}1&0\\0&0\end{bmatrix}, X\right\rangle$$
$$\text{s.t.} \ \left\langle \begin{bmatrix}0&1\\0&0\end{bmatrix}, X\right\rangle - 1 \in \{0\}$$
$$X\in S_+^n$$

$\Downarrow$

$$\min_{x,y} \ x$$
$$\text{s.t.} \ \ x\geq 0$$
$$xy\geq 1$$



the optimal value is $p^*=0$, but cannot be attained cones – see Definition 4.5.5. If a linear program is infeasible, then it will remain infeasible under any sufficiently small perturbation of the right-hand side **b**. In contrast, there are infeasible cone programs that become feasible under an *arbitrarily* small perturbation of **b**.

$A(x)$ may not be a closed convex cone, introducing the problem of closeness

A cone program is limit-feasible if

$$\begin{cases} \exists \{x_k \ \ k=1,\dots,+\infty\} \text{ and } \{u_k \ \ k=1,\dots,+\infty\} \text{ s.t. } x_k \in K,\ u_k \in L \\ \lim_{k\to+\infty} \{A(x_k) + u_k\} = b \end{cases}$$

$\{x_k\}$ and $\{u_k\}$ are called feasible sequences

eg. $\min_{x,y,z} \ 0$

s.t. $A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = b \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$b = [0, 1]$

$(x, y, z) \in \triangleleft$

is limit feasible, but not feasible

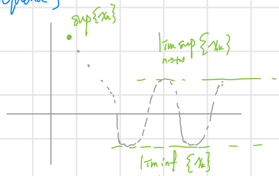
Every feasible cone program is limit-feasible

A limit-feasible cone program is feasible only if $C = \{A(x) + u \mid x \in K,\ u \in L\}$ is closed

Given a feasible sequence $\{x_k \ \ k=1,\dots,+\infty\}$ of a cone program,

the value of the feasible sequence is $\lim_{k\to+\infty} \inf \langle c, x_k \rangle$

the limit value of the problem is $\inf_{\{x_k\}} \left\{ \lim_{k\to+\infty} \inf \langle c, x_k \rangle \mid \{x_k\} \text{ is a feasible sequence} \right\}$



$V_n = \sup\{x_t \mid t > n\}$

$\lim_{n\to\infty} \sup x_n = \lim_{n\to\infty} V_n = \lim_{n\to\infty} \sup\{x_t \mid t > n\}$

• Value and Limit-Value

eg. $\min_{x,z} \ -z$

s.t. $(x, 0, z) \in \triangleleft$

$$\Rightarrow \quad \min_X \left\langle \begin{bmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{bmatrix}, X \right\rangle$$

$$\text{s.t. } \left\langle \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, X \right\rangle = 0$$

$$X \in S_+^n$$

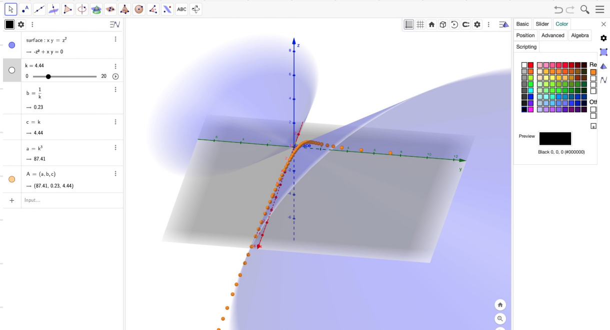
the problem is feasible when $x \geq 0$ $z = 0$, $\therefore$ the optimal value is 0

define $X_k = \begin{bmatrix} k^3 & k \\ k & 1/k \end{bmatrix} \in S_+^n$

$$\lim_{k\to+\infty} \langle A, X_k \rangle = \left\langle \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} k^3 & k \\ k & 1/k \end{bmatrix} \right\rangle = \lim_{k\to+\infty} 1/k = 0$$

$$\lim_{k\to\infty} \sup \left\langle \begin{bmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{bmatrix}, X_k \right\rangle = -\infty$$

the limit value is $-\infty$



An interior point of cone program $b-A(x)\in L$ $x\in K$ is a point x s.t.

$$\begin{cases} x\in \text{int}(K) \\ Ax=b \end{cases} \quad \text{if } L=\{0\} \qquad\qquad \begin{cases} x\in \text{int}(K) \\ b-A(x)\in \text{int}(L) \end{cases} \quad \text{if } L\neq\{0\}$$

if the cone program has an interior point, then the value equals the limit value

let γ' be the limit value, $\varepsilon>0$ be any real number

choose $\{x_k\}\{u_k\}$ s.t. $x_k\in K$, $u_k\in L$ $\lim_{k\to+\infty} A(x_k)+u_k=b$ $\quad \lim_{k\to+\infty}\inf\langle c,x_k\rangle=\gamma'$

let $\tilde{x}$ be an interior point

$\exists\{w_k\}$ s.t. $w_k\in K$, $b-A(w_k)\in L$ for large k $\|x_k-w_k\|_2\le\delta$ for arbitrarily small δ

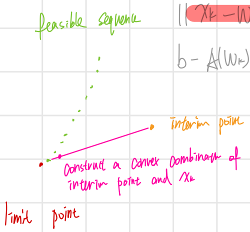
$$\|\langle c,x_k\rangle-\langle c,w_k\rangle\| = \|\langle c,(x_k-w_k)\rangle\| \le \|c\|_2\cdot\|x_k-w_k\|_2 \le \|c\|_2\,\delta$$

$$\therefore \lim_{k\to\infty}\inf\langle c,w_k\rangle < \lim_{k\to+\infty}\inf\langle c,x_k\rangle-\|c\|_2\,\delta = \gamma'+\|c\|_2\,\delta$$

$\therefore$ there are feasible solution of value arbitrarily close to the limit value

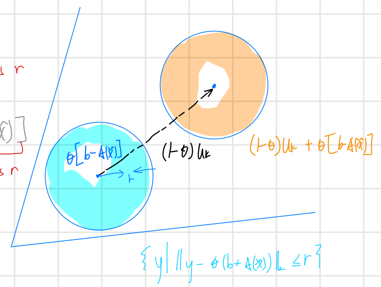
1° $L\neq\{0\}$

define $w_k=(1-\theta)x_k+\theta\tilde{x}$, $w_k\in K$



$\|x_k-w_k\|\le\delta$ by setting θ to be very small

$$b-A(w_k) = b-(1-\theta)A(x_k)-\theta A(\tilde{x})$$
$$= (1-\theta)[b-A(x_k)]+\theta[b-A(\tilde{x})]$$
$$= (1-\theta)[\underbrace{b-A(x_k)-u_k}_{\to 0}+\underbrace{u_k}_{\in L}]+\underbrace{\theta[b-A(\tilde{x})]}_{\in \text{int}(L) \text{ for radius } r}$$
$$= (1-\theta)[b-A(x_k)-u_k]+\underbrace{(1-\theta)u_k+\theta[b-A(\tilde{x})]}_{\in \text{int}(L) \text{ for radius } r}$$



$\therefore$ for large k, $b-A(w_k)\in L$

2° $L=\{0\}$

$K\subseteq V$ $L\subseteq W$ are finite dimension

pick a group basis of V $\{v_1\cdots v_n\}$

WLOG, assume $A(v_1)\cdots A(v_m)$ span the image of A

let $V'=\text{span}(v_1\cdots v_m)$

let $A^{-1}: \text{Im}(A)\to V'$

define $w_k = (1-\theta)\left[x_k + \underbrace{A^{-1}(\underbrace{b - A(x_k)}_{\in Im(A)})}_{\in V}\right] + \theta \tilde{x}$

$A w_k = (1-\theta)\left[A x_k + AA^{-1}(b - A(x_k))\right] + \theta A(\tilde{x})$

$= (1-\theta) b + \theta b$

$= b$

$w_k = \underbrace{\underbrace{(1-\theta) x_k + \underbrace{(1-\theta) A^{-1}(b - A(x_k)) + \underbrace{\theta \tilde{x}}_{\in K \text{ for radius } r}}_{\in K \text{ for large } k}}_{\in K \text{ for large } k}}$

$\|w_k - x_k\|_2 \le \delta$ for arbitrarily small δ by picking small θ

· Duality of Programming

$$\min_x \langle C, X\rangle$$

s.t. $b - A(x) \in L$: dual variable V

$x \in K$: dual variable Λ

$\min_x \langle C, X\rangle$

s.t. $A(x) - b \le_L 0$

$-x \le_K 0$

if $\Lambda \in K^*$, $V \in L^*$, then the dual function is a lower bound on p^*

$$g(\Lambda, V) = \inf_x \left[\langle C, X\rangle - \langle V, b - A(x)\rangle - \langle \Lambda, X\rangle\right]$$

$$\le \langle C, X^*\rangle - \langle V, b - A(x^*)\rangle - \langle \Lambda, X^*\rangle$$

$$\le p^*$$

$$g(\Lambda, V) = \inf_x \left[\langle C, X\rangle + \langle A^T(V), X\rangle - \langle V, b\rangle - \langle \Lambda, X\rangle\right]$$

$$= \inf_x \left[\langle C + A^T(V) - \Lambda, X\rangle - \langle V, b\rangle\right]$$

$$= \begin{cases} -\infty & C + A^T(V) - \Lambda \neq 0 \\ -\langle V, b\rangle & C + A^T(V) - \Lambda = 0 \end{cases}$$

the dual problem

$$\max \; -\langle V, b\rangle$$

$V \in L^*$

$C + A^T(V) \in K^*$

- Weak Duality limit value of P $\geq$ value of D

if the dual problem is feasible and the primal problem is limit-feasible
then the limit value of P is bounded by value of D

let $\{x_k\}$ $\{u_k\}$ be feasible sequence of P (i.e. $x_k \in K$, $u_k \in L$ $\lim_{k\to\infty} A(x_k)+u_k = b$)
let v be feasible in D

$$0 \leq \langle \underbrace{c+A^T(v)}_{K^*}, \underbrace{x_k}_{K} \rangle + \langle \underbrace{v}_{L^*}, \underbrace{u_k}_{L} \rangle = \langle c, x_k \rangle + \langle v, A(x_k)+u_k \rangle$$

$$\lim_{k\to\infty} \sup \langle c, x_k \rangle \geq \lim_{k\to\infty} \sup -\langle v, A(x_k)+u_k \rangle = -\langle v, b \rangle$$

- Regular Duality limit value of P = Value of D

Farkas lemma:
$\{Ax=b,\ x\in K\}$ is lim-fea
xor
$\{A^T(y) \in K^*,\ \langle b,y\rangle < 0\}$ is feasible

+ weak duality

The dual problem is feasible, has finite val d^*
$\Updownarrow$
The primal problem is limit feasible, has finite val p^*

Also, $p^* = d^*$

1° $\Downarrow$

if D is feasible and optimal value $= d^*$
$\forall\ v$ s.t. $v \in L^*$, $c+A^T(v) \in K^*$, have $-\langle v,b\rangle \leq d^*$
then $\forall\ v, z>0$ s.t.
$v/z \in L^*$ $c+A^T(v/z) \in K^*$, have $-\langle v/z, b\rangle \leq d^*$
$\therefore \forall\ v, z>0$ s.t. $v\in L^*$, $zc+A^T(v) \in K^*$, have $-\langle v,b\rangle \leq z d^*$

$v\in L^*$
$A^T(v)+zc \in K^*$
$\Downarrow$
$-\langle b,v\rangle \leq z d^*$
for $z>0$

also $\forall v$ s.t. $v\in L^*$, $A^T(v)\in K$, then have $-\langle v,b\rangle \leq 0$

if $\exists\ \tilde{v} \in L^*$ $A^T(\tilde{v}) \in K$ $-\langle \tilde{v}, b\rangle > 0$
then $\forall\ v$ feasible for D,
$v+t\tilde{v} \in L$ $A^T(v+t\tilde{v}) = A^T(v) + tA^T(\tilde{v}) \in K^*$ $v+t\tilde{v}$ is feasible for D
$-\langle b, v+t\tilde{v}\rangle = -\langle b,v\rangle - t\langle b,\tilde{v}\rangle \rightsquigarrow -\infty$ as $t\to+\infty$
contradict that d^*

$v\in L^*$
$A^T(v)+zc \in K^*$
$\Downarrow$
$-\langle b,v\rangle \leq z d^*$
for $z\geq 0$

$\to v\in L^*$ $A^T(v)+zc \in K^*$ $z\geq 0$ $\Rightarrow -\langle b,v\rangle \leq z d^*$

$\therefore$ the system $\left\{\begin{array}{l} v\in L^* \quad A^T(v)+zc \in K^* \\ -\langle b,v\rangle > z d^* \end{array}\right\}$ is infeasible

$\therefore$ the system $\left\{ \begin{bmatrix} A^T & c \\ I & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v \\ z \end{bmatrix} \in \begin{bmatrix} K^* \\ L^* \\ R_+ \end{bmatrix} \quad \langle [v,z],[b,d^*] \rangle < 0 \right\}$ is infeasible

by Farkas lemma $\left\{ \begin{matrix} A(x)=b \\ x \in K \end{matrix} \right\}$ is lim-fea or $\left\{ \begin{matrix} A^T(v) \in K^* \\ \langle b,v \rangle < 0 \end{matrix} \right\}$ is feasible

the system $\left\{ \begin{bmatrix} A & I & 0 \\ C^T & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ u \\ q \end{bmatrix} = \begin{bmatrix} b \\ d^* \end{bmatrix} \quad \begin{bmatrix} x \\ u \\ q \end{bmatrix} \in \begin{bmatrix} K \\ L \\ R_+ \end{bmatrix} \right\}$ is lim-fea.

$C(z) = C \cdot z$

$\langle C(z), x \rangle = \langle C \cdot z, x \rangle = z \cdot \langle C, x \rangle = \langle z, \langle C, x \rangle \rangle$

$\langle C(z), x \rangle = \langle z, C^T(x) \rangle$ $\quad \therefore C^T(x) = \langle C, x \rangle$

$\Updownarrow$

$\exists \{x_k\} \{u_k\} \{q_k\}$ s.t. $x_k \in K$ $u_k \in L$ $q_k \in R_+$ $\underbrace{\lim_{k \to +\infty} A(x_k) + u_k = b}_{\text{lim-fea}}$ $\underbrace{\lim_{k \to +\infty} \langle c, x_k \rangle + q_k = d^*}_{\text{limit value} \leq d^*}$

$\therefore$ by weak duality, $\forall$ feasible sequence $\{x_k\}$, $\lim_{k \to \infty} \sup \langle c, x_k \rangle \geq d^*$

$\therefore \lim_{k \to \infty} \sup \langle c, x_k \rangle = d^*$

γ $\Uparrow$

if P is lim-fea and the finite value is p^*

$\exists \{x_k\} \{u_k\}$ s.t. $x_k \in K$ $u_k \in L$ $\lim_{k \to \infty} A(x_k) + u_k = b$ $\lim_{k \to \infty} \langle c, x \rangle = p^*$

Assume D is infeasible (we'll get a contradiction)

if D is infeasible, we have

$A^T(v) + zc \in K^*$, $v \in L^*$ $\Rightarrow z \leq 0$

for any $A^T(v) + zc \in K^*$ $v \in L^*$, $z > 0$, v/z is feasible for D

$A^T(v) + zc \in K^*$

$\therefore$ the system $\{ A^T(v) + zc \in K^* \quad v \in L^* \quad z > 0 \}$ is infeasible

$\therefore$ the system $\left\{ \begin{bmatrix} A^T & c \\ I & 0 \end{bmatrix} \begin{bmatrix} v \\ z \end{bmatrix} \in \begin{bmatrix} K^* \\ L^* \end{bmatrix}, \quad \langle [0,-1],[v,z] \rangle < 0 \right\}$ is infeasible by Farkas Lemma

the system $\left\{ \begin{bmatrix} A & I \\ C^T & 0 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} x \\ u \end{bmatrix} \in \begin{bmatrix} K \\ L \end{bmatrix} \right\}$ is lim-fea

$\therefore \exists \{\tilde{x}_k\}, \{\tilde{u}_k\}$ s.t. $\tilde{x}_k \in K$ $\tilde{u}_k \in L$ $\lim_{k\to+\infty} A(\tilde{x}_k) + \tilde{u}_k = 0$ $\lim_{x\to\infty} \langle C, \tilde{x}_k \rangle = -1$

consider the sequence $\{x_k + t\tilde{x}_k\}$ $\{u_k + t\tilde{u}_k\}$ $t \geq 0$

$x_k + t\tilde{x}_k \in K$ $\quad u_k + t\tilde{u}_k \in L$ $\quad \lim_{k\to+\infty} A(x_k + t\tilde{x}_k) + (u_k + t\tilde{u}_k) = b$

$\therefore \{x_k + t\tilde{x}_k\}$ $\{u_k + t\tilde{u}_k\}$ is feasible sequence

but $\lim_{t\to\infty} \langle C, x_k + t\tilde{x}_k \rangle = \lim_{t\to\infty} \langle C, x_k \rangle + t \lim_{t\to\infty} \langle C, \tilde{x}_k \rangle = p^* - t$

Contradict that P has limit value p^*

$\therefore$ when p is lim-fea, D must be fea.

D is feasible, we have proved that $d^* = p^*$ when D is feasible

· Strong Duality

Farkas Lemma, weak duality → Regular duality ($\lim p^* = d^*$)

Regular duality, $\exists$ interior point, $\lim p = p$ → Strong duality

if the primal problem is feasible, has a finite p^*, and has an interior point then the dual problem is feasible, and has $d^* = p^*$

$\because$ P is feasible
$\therefore \lim p^* = p^*$

$\because$ P is lim-fea.
$\therefore$ D is feasible and $\lim p^* = d^*$

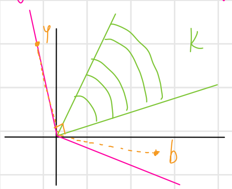
$p^* = d^*$

$\lim p^* = \inf_{\{x_k\}} \{ \lim_{k\to+\infty} \inf \langle C, x_k \rangle \mid \{x_k\} \text{ is feasible sequence} \}$

Separation

let $K \subseteq V$ be a close cone, $b \in V \setminus K$

$\exists\, y \in V$ s.t. $\langle y, x\rangle \geq 0 \ \forall x \in K$ and $\langle y, b\rangle < 0$



P: $\min_{x} \ \langle c, x\rangle$

s.t. $b - A(x) \in L$

$x \in K$

D: $\max_{v} \ -\langle b, v\rangle$

s.t. $v \in L^*$

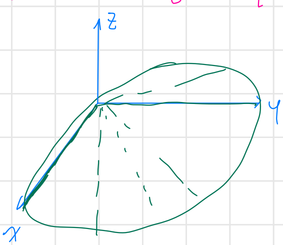
$A^T(v) + c \in K^*$

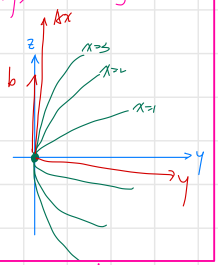
Farkas Lemma

$$\left[\begin{cases} Ax = b \\ x \geq 0 \end{cases} \text{feasible}\right] \text{ XOR } \left[\begin{cases} A^T y \geq 0 \\ \langle b, y\rangle < 0 \end{cases} \text{fea}\right]$$

Farkas Lemma Cone Version

$$\left[\begin{cases} A(x) = b \\ x \in K \end{cases} \text{lim-fea}\right] \text{ xor } \left[\begin{cases} A^T(y) \in K^* \\ \langle b, y\rangle < 0 \end{cases} \text{fea}\right]$$





Weak Duality

$$\left.\begin{array}{l} P \text{ lim-fea} \\ D \text{ fea} \end{array}\right\} \Longrightarrow \text{lim } p^* \geq d^*$$

Regular Duality

$$\left.\begin{array}{l} P \text{ lim-fea} \\ \text{finite lim } p^* \end{array}\right\} \qquad \begin{array}{l} D \text{ fea} \\ d^* = \text{lim } p^* \end{array}$$

$\exists$ interior point

$\text{lim } p^* = p^*$

Strong Duality

P feasible, finite value, interior point

$\Downarrow$

D feasible, $d^* = p^*$

Eg. Largest Eigen Value

$$\lambda_{max} = \max_x \{ x^T C x : \|x\|_2 = 1 \} \quad (C \in S^n)$$

$$\max_x \; x^T C x \quad \text{s.t. } x^T x = 1 \;\Rightarrow\; \min_x \langle -C, xx^T \rangle \quad \text{s.t. } tr(xx^T) = 1 \;\Rightarrow\; \min_X \langle -C, X \rangle \quad \text{s.t. } tr(X) = 1,\; X \succeq 0,\; (rank(X) = 1) \;\Rightarrow\; \min_X \langle -C, X \rangle \quad \text{s.t. } tr(X) = 1,\; X \succeq 0$$

relaxation

$\tilde{p}$ $\qquad$ p

p and $\tilde{p}$ has the same value

$\forall X$ feasible for p, $X = \sum_i \lambda_i \underbrace{u_i u_i^T}$ $\lambda_i \geq 0$

unit vector, trace 1

$tr(X) = \sum_i \lambda_i = 1$

$\langle -C, X \rangle = \langle -C, \sum_i \lambda_i u_i u_i^T \rangle = -\sum_i \lambda_i \langle C, u_i u_i^T \rangle \geq -\max_i \langle C, u_i u_i^T \rangle$

and $\tilde{X} = u_i u_i^T$ is feasible for $\tilde{p}$

$\therefore$ $\forall X$ feasible for p,

$\exists \tilde{X}$ feasible for $\tilde{p}$ with $\langle -C, X \rangle \geq -\langle C, \tilde{X} \rangle$

$\therefore$ $\tilde{p}^* \leq p^*$

$\because$ p is a relaxation for $\tilde{p}$ $\quad \tilde{p}^* \geq p^*$

$\therefore$ $\tilde{p}^* = p^*$

$$\min_X \langle -C, X \rangle \quad \text{s.t. } 1 - \langle I, X \rangle = 0,\; X \succeq 0 \;\Rightarrow\; \max_v \; -v \quad \text{s.t. } vI - C \succeq 0 \;\Rightarrow\; \min_v \; v \quad \text{s.t. } vI \succeq C$$

$\langle A^T(X), v \rangle = \langle \langle I, X \rangle, v \rangle = \langle X, vI \rangle = \langle X, A^T(v) \rangle$ $\qquad$ $d^* = v = \lambda_{max}(C)$