- Kronecker Product

the kronecker product of matrix $A \in \mathbb{R}^{m\times n}$ $B \in \mathbb{R}^{p\times q}$ is

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & & \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}$$

the vec-operator for matrix $A \in \mathbb{R}^{m\times n}$ is

$$\mathrm{vec}(A) = (a_{11} \cdots a_{m1}, a_{12} \cdots a_{m2}, \cdots, a_{1n} \cdots a_{mn})$$

$$\mathrm{tr}(A^T B) = \langle A, B \rangle = \langle \mathrm{vec}(A), \mathrm{vec}(B) \rangle$$

- Properties of Kronecker Product

$$(\alpha A) \otimes B = A \otimes (\alpha B) = \alpha (A \otimes B)$$

$$(A \otimes B)^T = A^T \otimes B^T$$

$$(A + B) \otimes C = (A \otimes C) + (B \otimes C)$$

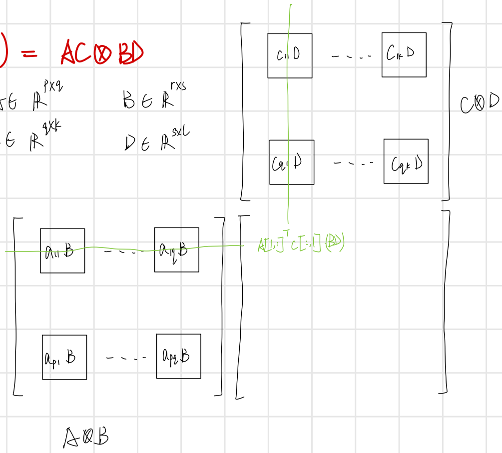
$$A \otimes (B + C) = (A \otimes B) + (A \otimes C)$$

$$(A \otimes B) \otimes C = A \otimes (B \otimes C)$$

$$(A \otimes B)(C \otimes D) = AC \otimes BD$$

$\forall A \in \mathbb{R}^{p\times q}$ $B \in \mathbb{R}^{r\times s}$

$C \in \mathbb{R}^{q\times k}$ $D \in \mathbb{R}^{s\times l}$



$$\mathrm{tr}(A \otimes B) = \mathrm{tr}(A)\,\mathrm{tr}(B) = \mathrm{tr}(B \otimes A)$$

$$\det(A \otimes B) = \det(A)^n \det(B)^m \quad \forall A \in \mathbb{R}^{m\times m} \; B \in \mathbb{R}^{n\times n}$$

$A \otimes B$ is non-singular iff A and B are non-singular

$$A \otimes B = (A I_m) \otimes (I B_n) = (A \otimes I_n)(I_m \otimes B)$$

$$\det(A \otimes B) = \det(A \otimes I_n \cdot \det(I_m \otimes B) = \det\left(\begin{bmatrix} a_{11} I & \cdots & a_{1m} I \\ \vdots & \ddots & \\ a_{m1} I & & a_{mm} I \end{bmatrix} \right) \cdot \det\left(\begin{bmatrix} \boxed{B} & & \\ & \boxed{B} & \\ & & \ddots & \boxed{B} \end{bmatrix} \right)$$

$$= \det(A)^n \cdot \det(B)^m$$

if $A \in \mathbb{R}^{m \times m}$, $B \in \mathbb{R}^{n \times n}$ are non-singular, $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$

$$(A \otimes B)(A^{-1} \otimes B^{-1}) = (A A^{-1}) \otimes (B B^{-1}) = I$$

- Kronecker Sum

the kronecker sum of $A\in \mathbb{R}^{m\times m}$, $B\in \mathbb{R}^{n\times n}$ is

$$A\oplus B=(I_n\otimes A)+(B\otimes I_m)$$

$$\underbrace{\begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}}_{I_n\otimes A}+\underbrace{\begin{bmatrix} b_{11}I & b_{12}I \\ b_{21}I & b_{22}I \end{bmatrix}}_{B\otimes I_m}=$$

let $A\in S^m$ $B\in S^n$. $\lambda\in\sigma(A)$ with eigenvector x

$\mu\in\sigma(B)$ with eigenvector y

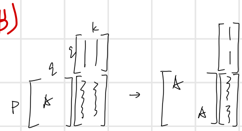
then $\lambda+\mu$ is an eigenvalue of $A\oplus B$, with eigenvector $y\otimes x$

$$\begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}\begin{bmatrix} y_1x \\ y_2x \end{bmatrix}+\begin{bmatrix} b_{11}I & b_{12}I \\ b_{21}I & b_{22}I \end{bmatrix}\begin{bmatrix} y_1x \\ y_2x \end{bmatrix}=\begin{bmatrix} y_1\lambda x \\ y_2\lambda x \end{bmatrix}+By\otimes x=y\otimes\lambda x+\mu y\otimes x=(\lambda+\mu)\cdot(y\otimes x)$$

- Matrix Equation and Kronecker Product

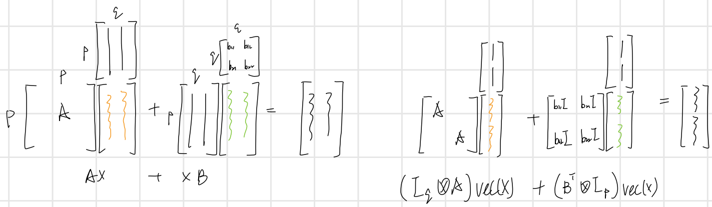
$AX=B \Rightarrow (I_k\otimes A)\cdot vec(X)=vec(B)$

$A\in \mathbb{R}^{p\times q}$ $X\in \mathbb{R}^{q\times k}$



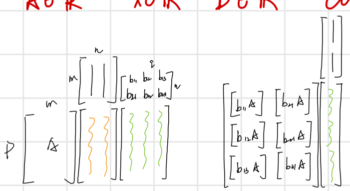
$AX+XB=C \Rightarrow (I_q\otimes A)vec(X)+(B^T\otimes I_p)vec(X)=vec(C) \Rightarrow (A\oplus B^T)vec(X)=vec(C)$

$A\in \mathbb{R}^{p\times p}$ $X\in \mathbb{R}^{p\times q}$ $B\in \mathbb{R}^{q\times q}$ $C\in \mathbb{R}^{p\times q}$



$AXB=C \Rightarrow (B^T\otimes A)vec(X)=vec(C)$

$A\in \mathbb{R}^{p\times m}$ $X\in \mathbb{R}^{m\times n}$ $B\in \mathbb{R}^{n\times q}$ $C\in \mathbb{R}^{p\times q}$



· The Symmetric Kronecker Product

for any $S \in S^n$, $\operatorname{svec}(S) = (S_{11}, \sqrt{2} S_{21}, \cdots \sqrt{2} S_{n1}, \; S_{22}, \sqrt{2} S_{32} \; \cdots)$

$$\operatorname{vec}\left(\begin{bmatrix} S_{11} & & \\ S_{21} & S_{22} & \\ \vdots & & \ddots \\ S_{n1} & \cdots & S_{nn} \end{bmatrix} .* \begin{bmatrix} 1 & & \\ \sqrt{2} & 1 & \\ \vdots & & \ddots \\ \sqrt{2} & \cdots & 1 \end{bmatrix} \right)$$

$$\operatorname{tr}(ST) = \langle S, T \rangle = \operatorname{svec}(S)^T \operatorname{svec}(T) \qquad \forall S \in S^n \; T \in S^n$$

$$\exists \; QQ^T = I \quad \text{s.t.} \quad Q \operatorname{vec}(S) = \operatorname{svec}(S) \qquad Q^T \operatorname{svec}(S) = \operatorname{vec}(S)$$

$$Q = \begin{bmatrix} - \operatorname{vec}\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) - \\ - \operatorname{vec}\left(\begin{bmatrix} 0 & \frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) - \\ - \operatorname{vec}\left(\begin{bmatrix} 0 & 0 & \frac{\sqrt{2}}{2} \\ 0 & 0 & 0 \\ \frac{\sqrt{2}}{2} & 0 & 0 \end{bmatrix}\right) - \\ - \operatorname{vec}\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) - \\ \vdots \end{bmatrix} \quad \frac{n(n+1)}{2}$$

$$\xrightarrow{\quad n^2 \quad}$$

$$Q^T Q \operatorname{vec}(S) = Q^T \operatorname{svec}(S)$$
$$= \operatorname{vec}(S)$$

($Q^T Q$ is identity on S^n)

let $Q \in \mathbb{R}^{[\frac{1}{2}(n+1)n \times n^2]}$ be a mapping matrix for S^n
the symmetric kronecker product is defined as

$$G \otimes_s H = \frac{1}{2} Q (G \otimes H + H \otimes G) Q^T \qquad \forall G, H \in \mathbb{R}^{n \times n}$$

$$\begin{aligned} (G \otimes_s H) \operatorname{svec}(U) &= \frac{1}{2} Q (G \otimes H + H \otimes G) Q^T \operatorname{svec}(U) \\ &= \frac{1}{2} Q (G \otimes H + H \otimes G) \operatorname{vec}(U) \\ &= \frac{1}{2} Q \left[(G \otimes H) \operatorname{vec}(U) + (H \otimes G) \operatorname{vec}(U) \right] \\ &= \frac{1}{2} Q \left[\operatorname{vec}(HUG^T) + \operatorname{vec}(GUH^T) \right] \\ &= \frac{1}{2} \operatorname{svec}(HUG^T + GUH^T) \end{aligned}$$

$$(G \otimes_s H) \operatorname{svec}(U) = \frac{1}{2} \operatorname{svec}(HUG^T + (HUG)^T) \qquad \text{if } U \in S^n$$

- Properties of Symmetric Kronecker Product

$$(\alpha A) \otimes_s B = A \otimes_s (\alpha B) = \alpha \cdot (A \otimes_s B) \quad \forall A, B \in \mathbb{R}^{n \times n}$$

$$(A \otimes_s B)^T = A^T \otimes_s B^T$$

$$
\begin{aligned}
(A \otimes_s B)^T &= \frac{1}{2}\left[Q\left[A \otimes B + B \otimes A\right] Q^T\right]^T \\
&= \frac{1}{2} Q \left[A \otimes B + B \otimes A\right]^T Q \\
&= \frac{1}{2} Q \left[(A \otimes B)^T + (B \otimes A)^T\right] Q \\
&= \frac{1}{2} Q \left[A^T \otimes B^T + B^T \otimes A^T\right] Q \\
&= (A^T \otimes_s B^T)
\end{aligned}
$$

$A \otimes_s I$ is symmetric iff A is symmetric

$$(A+B) \otimes_s C = A \otimes_s C + B \otimes_s C \quad \forall A \; B \; C$$

$$A \otimes_s (B+C) = A \otimes_s B + A \otimes_s C$$

$$
\begin{aligned}
(A+B) \otimes_s C &= \frac{1}{2} Q\left[(A+B) \otimes C + C \otimes (A+B)\right] Q^T \\
&= \frac{1}{2} Q\left[A \otimes C + B \otimes C + C \otimes A + C \otimes B\right] Q^T \\
&= \frac{1}{2} Q\left[A \otimes C + C \otimes A\right] Q^T + \frac{1}{2} Q\left[B \otimes C + C \otimes B\right] Q^T \\
&= A \otimes_s C + B \otimes_s C
\end{aligned}
$$

$$(A \otimes_s B)(C \otimes_s D) = \frac{1}{2}\left[AC \otimes_s BD + AD \otimes_s BC\right] \quad \forall A, B, C, D \in \mathbb{R}^{n \times n}$$

$$
\begin{aligned}
(A \otimes_s B)(C \otimes_s D) &= \frac{1}{2} Q\left[A \otimes B + B \otimes A\right] Q^T \cdot \frac{1}{2} Q\left[C \otimes D + D \otimes C\right] Q^T \\
&= \frac{1}{4} Q \left(A \otimes B + B \otimes A\right)\left(C \otimes D + D \otimes C\right) Q^T \\
&= \frac{1}{4} Q \left[(A \otimes B)(C \otimes D) + (A \otimes B)(D \otimes C) + (B \otimes A)(C \otimes D) + (B \otimes A)(D \otimes C)\right] Q^T \\
&= \frac{1}{4} Q \left[AC \otimes BD + AD \otimes BC + BC \otimes AD + BD \otimes AC\right] Q^T \\
&= \frac{1}{4} Q\left[AC \otimes BD + BD \otimes AC\right] Q^T + \frac{1}{4} Q\left[AD \otimes BC + BC \otimes AD\right] Q^T \\
&= \frac{1}{2}\left[AC \otimes_s BD + AD \otimes_s BC\right]
\end{aligned}
$$

$$(A \otimes_s A)^{-1} = A^{-1} \otimes_s A^{-1} \quad \forall A \in \mathbb{R}^{n \times n} \text{ is non-singular}$$

$$
\begin{aligned}
(A \otimes_s A)(A^{-1} \otimes_s A^{-1}) &= \frac{1}{2}\left[AA^{-1} \otimes_s AA^{-1} + AA^{-1} \otimes_s AA^{-1}\right] \\
&= \frac{1}{2} I \otimes_s I \\
&= \frac{1}{2} Q\left[I \otimes I + I \otimes I\right] Q^T \\
&= I
\end{aligned}
$$

$(A \otimes_s B)^{-1} \neq A^{-1} \otimes_s B^{-1}$ in general

- Sloving for AHO Direction

$$\begin{cases} \mathcal{A}(X)=b \\ C-\Lambda+\mathcal{A}^T(v)=0 \\ X\Lambda=\frac{1}{t}I \end{cases} \Rightarrow \begin{cases} \mathcal{A}(X)=b \\ C-\Lambda+\mathcal{A}^T(v)=0 \\ \frac{1}{2}(X\Lambda+\Lambda X)=\frac{1}{t}I \end{cases}$$

$\frac{1}{t}I-X\Lambda$ may not be symmetric

$$\Rightarrow \begin{cases} \mathcal{A}(\Delta X)= b-\mathcal{A}(X) & \} -r_p \\ -\Delta\Lambda+\mathcal{A}^T(\Delta v) = -C+\Lambda-\mathcal{A}^T(\Delta v) & \} -r_d \\ \frac{1}{2}(\Delta X\Lambda+\Lambda\Delta X+X\Delta\Lambda+\Delta\Lambda X)=\frac{1}{t}I-\frac{1}{2}(X\Lambda+\Lambda X) & \} -r_c \end{cases}$$

$$\begin{bmatrix} 0 & \mathcal{A}(\cdot) & 0 \\ \mathcal{A}^T(\cdot) & 0 & -I \\ 0 & \mathcal{E}_\Lambda(\cdot) & \mathcal{F}_X(\cdot) \end{bmatrix} \begin{bmatrix} \Delta v \\ \Delta X \\ \Delta\Lambda \end{bmatrix} = \begin{bmatrix} -r_p \\ -r_d \\ -r_c \end{bmatrix}$$

$$\mathcal{E}_\Lambda(\Delta X)=\frac{1}{2}(\Delta X\Lambda+\Lambda\Delta X)$$
$$\mathcal{F}_X(\Delta\Lambda)=\frac{1}{2}(X\Delta\Lambda+\Delta\Lambda X)$$

$$AX+XB=C \quad \Rightarrow \left(I\otimes A+B^T\otimes I\right)vec(X)=vec(C)$$

$$\begin{bmatrix} A & \\ & A \end{bmatrix}\begin{bmatrix} | \\ | \end{bmatrix} + \begin{bmatrix} B_{11}I & B_{21}I \\ B_{12}I & B_{22}I \end{bmatrix}\begin{bmatrix} | \\ | \end{bmatrix}$$

$$\mathcal{E}_\Lambda(\Delta X)+\mathcal{F}_X(\Delta\Lambda)=\frac{1}{2}\left(\Lambda\Delta X+\Delta X\Lambda\right)+\frac{1}{2}\left(X\Delta\Lambda+\Delta\Lambda X\right)=-r_c$$

$$\frac{1}{2}\left(I\otimes\Lambda+\Lambda\otimes I\right)vec(\Delta X)+\frac{1}{2}\left(I\otimes X+X\otimes I\right)vec(\Delta\Lambda)=-vec(r_c)$$

$$\frac{1}{2}Q\left(I\otimes\Lambda+\Lambda\otimes I\right)Q^TQ\,vec(\Delta X)+\frac{1}{2}Q\left(I\otimes X+X\otimes I\right)Q^TQ\,vec(\Delta\Lambda)=-Q\,vec(r_c)$$
$$\left(I\otimes_s\Lambda\right)svec(\Delta X)+\left(I\otimes_s X\right)svec(\Delta\Lambda)=svec(-r_c)$$

$$\mathcal{A}^T(\Delta v)-\Delta\Lambda=-r_d$$
$$vec\left(\mathcal{A}^T(\Delta v)\right)-vec(\Delta\Lambda)=vec(-r_d)$$
$$Q\,vec\left(\mathcal{A}^T(\Delta v)\right)-svec(\Delta\Lambda)=svec(-r_d)$$
$$\sum_i \Delta v_i\, svec(A_i)-svec(\Delta\Lambda)=svec(-r_d)$$

$$\sum_i \Delta v_i\, svec(A_i)-svec(\Delta\Lambda)=svec(-r_d)$$

$A(\Delta X) = -r_p$

$$\langle A_i, \Delta X\rangle = -r_{p_i}$$

$$\langle vec(A_i), vec(\Delta X)\rangle = -r_{p_i}$$

$$\langle svec(A_i), svec(\Delta X)\rangle = -r_{p_i}$$

$$\langle svec(A_i), svec(\Delta X)\rangle = -r_{p_i}$$

svec system:

$$\begin{bmatrix} 0 & svec(A)^T & 0 \\ svec(A) & 0 & -I \\ 0 & I \otimes_s \Lambda & I \otimes_s X \end{bmatrix} \begin{bmatrix} \Delta v \\ svec(\Delta X) \\ svec(\Delta \Lambda) \end{bmatrix} = - \begin{bmatrix} r_p \\ svec(r_d) \\ svec(r_c) \end{bmatrix}$$

$$r_{p_i} = \langle A_i, X\rangle = \langle svec(A_i), svec(X)\rangle$$

$$svec(r_d) = svec(C + A^T(v) - \Lambda)$$
$$= svec(C) + \sum_i v_i \, svec(A_i) - svec(\Lambda)$$

row3 + $(I \otimes_s X)$ row2 :

$$row3' = [(I \otimes_s X) svec(A), \; I \otimes_s \Lambda, \; 0]$$

$$row1 - (I \otimes_s \Lambda)^{-1} svec(A)^T [\,row3 + I \otimes_s X \; row2\,]$$

$$-(I \otimes_s \Lambda)^{-1} svec(A)^T (I \otimes_s X) svec(A) \Delta v =$$

$$-r_p - (I \otimes_s \Lambda)^{-1} svec(A)^T \left(-svec(r_c) - (I \otimes_s X) svec(r_d)\right)$$

$$svec(\Delta \Lambda) = svec(A)\Delta v + svec(r_d)$$

$$svec(\Delta X) = (I \otimes_s \Lambda)^{-1} [-svec(r_c) - (I \otimes_s X) svec(\Delta \Lambda)]$$