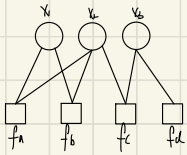
Factor Graph

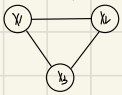
$p(x) = \frac{1}{Z} \prod_s f_s(x_s)$

Each factor f_s is a function of corresponding x_s

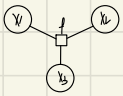
each potential has its own factor node

Factor graphs are always bipartite

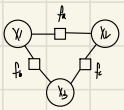




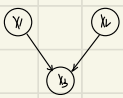
$p(x_1, x_2, x_3)$



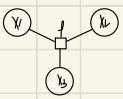
$p(x_1, x_2, x_3) = f(x_1, x_2, x_3)$



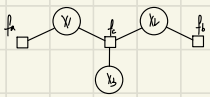
$p(x_1, x_2, x_3) = f_a(x_1, x_2) f_b(x_1, x_3) f_c(x_2, x_3)$



$p(x_1, x_2, x_3)$



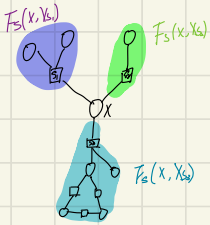
$p(x_1, x_2, x_3) = f(x_1, x_2, x_3)$



$p(x_1, x_2, x_3) = f_a(x_1) \cdot f_b(x_2) \cdot f_c(x_1, x_2, x_3)$

Message Passing

Substitute $p(x)$ with factor graph representation and interchange summations and products



partition the factors in the joint distribution into groups,

each group is associated with each of the factor nodes that's a neighbor of x

$$p(x) = \sum_{\setminus x} \prod_{s \in ne(x)} F_s(x, x_s)$$

where $ne(x)$ is the set of factor nodes that are neighbor of x

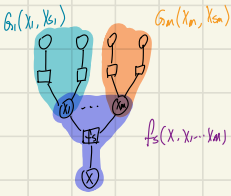
x_s is the set of variable in the subtree that's connected to x via factor node f_s

$F_s(x, x_s)$ is the product of all factors in the set associated with f_s

$\mu_{f_s \to x}(x) = \sum_{x_s} F_s(x, x_s)$ Message from Factor to Variable

$$p(x) = \sum_{x_1} \prod_{s \in ne(x)} F_s(x, x_s) = \prod_{s \in ne(x)} \sum_{x_s} F_s(x, x_s) = \prod_{s \in ne(x)} \mu_{f_s \to x}(x)$$

the marginal is given by product of all incoming messages to x



$F_s(x, x_s) = f_s(x, x_1, \cdots x_M) \cdot G_1(x_1, x_{s_1}) \cdots G_M(x_M, x_{s_M})$

$$\mu_{f_s \to x}(x) = \sum_{x_s} F_s(x, x_s) = \sum_{x_1} \cdots \sum_{x_M} \sum_{x_{s_1}} \cdots \sum_{x_{s_M}} f_s(x, x_1, \cdots x_M) \cdot G_1(x_1, x_{s_1}) \cdots G_M(x_M, x_{s_M})$$

$$= \sum_{x_1} \cdots \sum_{x_M} f_s(x, x_1, \cdots x_M) \cdot \sum_{x_{s_1}} \cdots \sum_{x_{s_M}} G_1(x_1, x_{s_1}) \cdots G_M(x_M, x_{s_M})$$

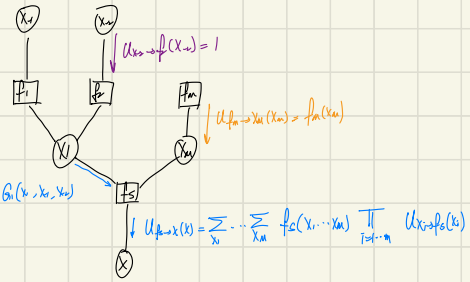
$$= \sum_{x_1} \cdots \sum_{x_M} f_s(x, x_1, \cdots x_M) \cdot \prod_{m \in ne(f_s) \setminus x} \sum_{x_{s_m}} G_m(x_m, x_{s_m})$$

$$= \sum_{x_1} \cdots \sum_{x_M} f_s(x, x_1, \cdots x_M) \cdot \prod_{m \in ne(f_s) \setminus x} \mu_{x_m \to f_s}(x_m)$$

$$\mu_{x_m \to f_s}(x_m) = \sum_{\sim x_m} G_{x_m}(x_m, \dots)$$

$$= \prod_{l \in ne(x_m) \setminus f_s} \mu_{f_l \to x_m}(x_m)$$

Message from variable to factor



If the leaf node is a variable node

then $\mu_{x \to f}(x) = 1$ // initialize as uniform

If the leaf node is a factor node

then $\mu_{f \to x}(x) = f(x)$ // initialize with prior

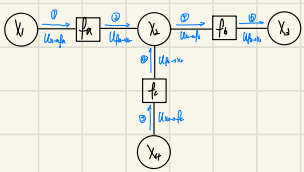
· Sum-Product Algorithm

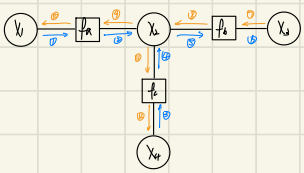
pick an arbitrary node as root

Compute and propagate message from the leaf node to the root, store received messages

Compute and propagate message from the root node to the leaves, store received message

Compute the product of received message at each node





① $\mu_{x_1 \to f_a}(x_1) = 1$ // Uniform

② $\mu_{f_a \to x_2}(x_2) = \sum_{x_1} f_a(x_1, x_2) \cdot \mu_{x_1 \to f_a}(x_1) = \sum_{x_1} f_a(x_1, x_2)$

③ $\mu_{x_4 \to f_c}(x_4) = 1$ // Uniform

④ $\mu_{f_c \to x_2}(x_2) = \sum_{x_4} f_c(x_2, x_4) \cdot \mu_{x_4 \to f_c}(x_4) = \sum_{x_4} f_c(x_2, x_4)$

⑤ $\mu_{x_2 \to f_b}(x_2) = \mu_{f_a \to x_2}(x_2) \cdot \mu_{f_c \to x_2}(x_2)$

⑥ $\mu_{f_b \to x_3}(x_3) = \sum_{x_2} f_b(x_2, x_3) \cdot \mu_{x_2 \to f_b}(x_2)$

⑦ $\mu_{x_3 \to f_b}(x_3) = 1$ // Uniform

⑧ $\mu_{f_b \to x_2}(x_2) = \sum_{x_3} f_b(x_2, x_3) \cdot \mu_{x_3 \to f_b}(x_3) = \sum_{x_3} f_b(x_2, x_3)$

⑨ $\mu_{x_2 \to f_a}(x_2) = \mu_{f_c \to x_2}(x_2) \cdot \mu_{f_b \to x_2}(x_2)$

⑩ $\mu_{f_a \to x_1}(x_1) = \sum_{x_2} f_a(x_1, x_2) \cdot \mu_{x_2 \to f_a}(x_2)$

⑪ $\mu_{x_4 \to f_c}(x_4) = \mu_{f_b \to x_3}(x_3) \cdot \mu_{f_d \to x_4}(x_4)$

⑫ $\mu_{f_c \to x_2}(x_2) = \sum_{x_4} f_c(x_2, x_4) \cdot \mu_{x_4 \to f_c}(x_4)$